Stochastic Local Volatility in QuantLib

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Heston Stochastic Local Volatility

Fokker-Planck Equations
- Square Root Process
- Boundary Conditions
- Coordinate and Density Transformations

Calibration
Local Volatility [Dupire 1994]

- Local Volatility $\sigma_{LV}(S, t)$ as function of spot level $S_t$ and time $t$:

$$d \ln S_t = \left( r_t - q_t - \frac{1}{2} \sigma_{LV}^2(S, t) \right) dt + \sigma_{LV}(S, t) dW_t$$

$$\sigma_{LV}^2(S, t) = \left. \frac{\partial C}{\partial T} + (r_t - q_t) K \frac{\partial C}{\partial K} + q_tC \right|_{K=S, T=t}$$

- Consistent with option market prices.
- Model is often criticized for its unrealistic volatility dynamics.
- Dupire formula is mathematically appealing but also unstable.
Stochastic Volatility [Heston 1993]

- Stochastic volatility given by a square-root process:

\[
\begin{align*}
    d \ln S_t &= \left( r_t - q_t - \frac{1}{2} \nu_t \right) dt + \sqrt{\nu_t} dW_t^S \\
    d\nu_t &= \kappa (\theta - \nu_t) dt + \sigma \sqrt{\nu_t} dW_t^\nu \\
    \rho dt &= dW_t^\nu dW_t^S
\end{align*}
\]

- Semi-analytical solution for European call option prices:

\[
C(S_0, K, \nu_0, T) = SP_1 - Ke^{-(r_t-q_t)T} P_2
\]

\[
P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \Re \left[ e^{-iu \ln K} \phi_j(S_0, K, \nu_0, T, u) \right] du
\]

- More realistic volatility dynamics.
- Does often not exhibit enough skew for short dated expiries.
Example: Differences in $\delta$ and $\gamma$

The implied and local volatility surface is derived from the Heston model and therefore the option prices between all models match. $S_0 = 5000$, $\kappa = 5.66$, $\theta = 0.075$, $\sigma = 1.16$, $\rho = -0.51$, $\nu_0 = 0.19$, $T = 1.7$
Add leverage function $L(S_t, t)$ and mixing factor $\eta$:

$$d \ln S_t = \left( r_t - q_t - \frac{1}{2} L(S_t, t)^2 \nu_t \right) dt + L(S_t, t) \sqrt{\nu_t} dW_t^S$$

$$d \nu_t = \kappa (\theta - \nu_t) dt + \eta \sigma \sqrt{\nu_t} dW_t^\nu$$

$$\rho dt = dW_t^\nu dW_t^S$$

Leverage $L(x_t, t)$ is given by probability density $p(S_t, \nu, t)$ and

$$L(S_t, t) = \frac{\sigma_{LV}(S_t, t)}{\sqrt{\mathbb{E}[\nu_t | S = S_t]}} = \sigma_{LV}(S_t, t) \sqrt{\frac{\int_{\mathbb{R}^+} p(S_t, \nu, t) d\nu}{\int_{\mathbb{R}^+} \nu p(S_t, \nu, t) d\nu}}$$

Mixing factor $\eta$ tunes between stochastic and local volatility.
Cheat Sheet: Link between SDE and PDE

Starting point is a multidimensional SDE of the form:

\[ dx_t = \mu(x_t, t)\, dt + \sigma(x_t, t)\, dW_t \]

**Feynman-Kac**: price of a derivative \( u(x_t, t) \) with boundary condition \( u(x_T, T) \) at maturity \( T \) is given by:

\[
\frac{\partial}{\partial t} u + \sum_{k=1}^{n} \mu_i \frac{\partial}{\partial x_k} u + \frac{1}{2} \sum_{k,l=1}^{n} \left( \sigma \sigma^T \right)_{kl} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l} u - ru = 0
\]

**Fokker-Planck**: time evolution of the probability density function \( p(x_t, t) \) with the initial condition \( p(x, t = 0) = \delta(x - x_0) \) is given by:

\[
\frac{\partial}{\partial t} p = - \sum_{k=1}^{n} \frac{\partial}{\partial x_k} [\mu_i p] + \frac{1}{2} \sum_{k,l=1}^{n} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l} \left[ \left( \sigma \sigma^T \right)_{kl} p \right]
\]
The SLV model leads to following Feynman-Kac equation for a function $u : \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, $(x, \nu, t) \mapsto u(x, \nu, t)$:

$$
0 = \partial_t u + \frac{1}{2} L^2 \nu \partial_x^2 u + \frac{1}{2} \eta^2 \sigma^2 \nu \partial_\nu^2 u + \eta \sigma \rho L \partial_x \partial_\nu u + \left( r - q - \frac{1}{2} L^2 \nu \right) \partial_x u + \kappa (\theta - \nu) \partial_\nu u - ru
$$

- PDE can be solved using either Implicit scheme (slow) or more advanced operator splitting schemes like modified Craig-Sneyd or Hundsdorfer-Verwer in conjunction with damping steps (fast).
- Implementation is mostly harmless, extend FdmHestonOp.
The corresponding Fokker-Planck equation for the probability density $p : \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}, (x, \nu, t) \mapsto p(x, \nu, t)$ is:

$$
\partial_t p = \frac{1}{2} \partial_x^2 \left[ L^2 \nu p \right] + \frac{1}{2} \eta^2 \sigma^2 \partial^2_\nu \left[ \nu p \right] + \eta \sigma \rho \partial_x \partial_\nu \left[ L \nu p \right] \\
- \partial_x \left[ \left( r - q - \frac{1}{2} L^2 \nu \right) p \right] - \partial_\nu \left[ \kappa (\theta - \nu) p \right]
$$

- Numerical solution of the PDE is cumbersome due to difficult boundary conditions and the Dirac delta distribution as the initial condition.
- PDE can be efficiently solved using operator splitting schemes, preferable the modified Craig-Sneyd scheme.
Square Root Process

Main issues of the implementation are caused by the square root process:

\[ d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t}dW \]

It has the following Fokker-Planck equation for the probability density \( p : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, (\nu, t) \mapsto p(\nu, t) \):

\[ \partial_t p = \frac{\sigma^2}{2} \partial_{\nu}^2 [\nu p] - \partial_{\nu} [\kappa(\theta - \nu)p] \]

The stationary probability density \( \hat{p}(\nu) \) with \( \partial_t \hat{p}(\nu) = 0 \) is:

\[ \hat{p}(\nu) = \beta^\alpha \nu^{\alpha - 1} \exp(-\beta \nu) \Gamma(\alpha)^{-1}, \quad \alpha = \frac{2\kappa\theta}{\sigma^2}, \quad \beta = \frac{\alpha}{\theta} \]
Stationary Probability Density

\[
\lim_{\nu \to 0} \hat{p}(\nu) = \begin{cases} 
\infty & \text{if } \alpha < 1 \\
\theta^{-1} & \text{if } \alpha = 1 \\
0 & \text{if } \alpha > 1 
\end{cases}
\]

The square root process \( \nu_t \) is strictly positive if the Feller Condition \( \alpha > 1 \) is met.
The probability weight within $[\nu_{\text{min}}, \nu_{\text{max}}]$ of $p(\nu, t)$ is evolving by:

$$
\partial_t \int_{\nu_{\text{min}}}^{\nu_{\text{max}}} d\nu p = \int_{\nu_{\text{min}}}^{\nu_{\text{max}}} d\nu \left( \frac{\sigma^2}{2} \partial_\nu^2 [\nu p] - \partial_\nu [\kappa(\theta - \nu)p] \right)
$$

In order to avoid leaking of probability we enforce:

$$
\partial_t \int_{\nu_{\text{min}}}^{\nu_{\text{max}}} d\nu p = 0 \Rightarrow \left[ \frac{\sigma^2}{2} \partial_\nu [\nu p] - [\kappa(\theta - \nu)p] \right]_{\nu=\nu_{\text{min}}}^{\nu_{\text{max}}} = 0
$$

$$
\Rightarrow \left[ \frac{\sigma^2}{2} \partial_\nu [\nu p] - [\kappa(\theta - \nu)p] \right]_{\nu=\nu_{\text{min}}, \nu_{\text{max}}} = 0
$$

Zero Flux Boundary Condition
On a non-uniform grid \( \{ z_1, \ldots, z_n \} \) the two-sided approximation of \( \partial_z f \) is:

\[
\partial_z f(z_i) \approx \frac{h_{i-i}^2 f_{i+1} + (h_i^2 - h_{i-1}^2) f_i - h_i^2 f_{i-1}}{h_{i-1} h_i (h_{i-1} + h_i)}
\]

\[
= \frac{h_{i-1}}{h_{i-1} + h_i} \frac{f_{i+1} - f_i}{h_i} + \frac{h_i}{h_{i-1} + h_i} \frac{f_i - f_{i-1}}{h_{i-1}}
\]

With \( h_i := z_{i+1} - z_i \) and \( f_i := f(z_i) \). The second order derivative is approximated by:

\[
\partial_z^2 f(z_i) \approx \frac{h_{i-i} f_{i+1} - (h_{i-1} + h_i) f_i + h_i f_{i-1}}{\frac{1}{2} h_{i-1} h_i (h_{i-1} + h_i)}
\]
Sort by factors of $f_i$, set

\[
\begin{align*}
\zeta_i & := h_i h_{i-1} \\
\zeta_p & := h_i (h_{i-1} + h_i) \\
\zeta_m & := h_{i-1} (h_{i-1} + h_i)
\end{align*}
\]

then:

\[
\begin{align*}
\partial_z f(z_i) & \approx \frac{h_{i-1} f_{i+1}}{\zeta_i} + \frac{(h_i - h_{i-1}) f_i}{\zeta_i} - \frac{h_i f_{i-1}}{\zeta_m} \\
\partial_z^2 f(z_i) & \approx \frac{2}{\zeta_i} f_{i+1} - \frac{2}{\zeta_i} f_i + \frac{2}{\zeta_m} f_{i-1}
\end{align*}
\]
A general partial differential equation of the form

\[ \partial_t f = A(z) \partial^2_z f + B(z) \partial_z f + C(z) f \]

has therefore the spacial discretization:

\[ \partial_t f(z_i) = \frac{2A_i + B_i h_{i-1}}{\zeta_i^p} f_{i+1} + \left( \frac{-2A_i + B_i (h_i - h_{i-1})}{\zeta_i} + C_i \right) f_i \]

\[ + \frac{2A_i - B_i h_i}{\zeta_i^m} f_{i-1} \]

\[ =: \gamma_i f_{i+1} + \beta_i f_i + \alpha_i f_{i-1} \]
This is interpreted as a tridiagonal transfer matrix $T$ with diagonal $\beta_i$, upper diagonal $\gamma_i$, and lower diagonal $\alpha_i$:

$$
T := 
\begin{pmatrix}
\beta_1 & \gamma_1 & 0 & \ldots \\
\alpha_2 & \beta_2 & \gamma_2 & 0 & \ldots \\
0 & \alpha_3 & \beta_3 & \gamma_3 & 0 & \ldots \\
& & & & & \ddots \ddots \ddots \ddots \\
& & & & \alpha_{n-1} & \beta_{n-1} & \gamma_{n-1} \\
& & \vdots & \alpha_n & \beta_n
\end{pmatrix}
$$

Then

$$
\partial_t \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = T \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}
$$
Add $z_0$ below the lower boundary and $z_{n+1}$ above the upper boundary to the grid. The zero flux condition takes the general form

$$\left[ \partial_z A(z, t)f + B(z, t)f \right]_{z = z_0, z_{n+1}} \equiv 0$$

Lower Boundary: The partial derivative is discretized by a second order forward differentiation, so that all terms are given by grid points

$$\partial_z f(z_0) \approx - \frac{h_0^2 f_2 + (h_1 + h_0)^2 f_1 - ((h_1 + h_0)^2 - h_0^2) f_0}{h_0 h_1 (h_1 + h_0)}$$

$$= - \frac{h_0}{\zeta_p} f_2 + \left( \frac{h_0 + h_1}{\zeta_1} \right) f_1 - \left( \frac{2h_0 + h_1}{\zeta_1^m} \right) f_0$$
Boundary Condition

The general zero-flux boundary condition is therefore discretized at the lower boundary as

\[ 0 = -\frac{h_0}{\zeta_1} A_0 f_2 + \frac{(h_0 + h_1)}{\zeta_1} A_0 f_1 + \left( -\frac{(2h_0 + h_1)}{\zeta_1} A_0 + B_0 \right) f_0 \]

\[ = c_1 f_2 + b_1 f_1 + a_1 f_0 \]

\( \Rightarrow f_0 = -\frac{c_1}{a_1} f_2 - \frac{b_1}{a_1} f_1 \)

\[ \partial_t f_1 = \gamma_1 f_2 + \beta_1 f_1 + \alpha_1 f_0 \]

\[ = (\gamma_1 - \alpha_1 \frac{c_1}{a_1}) f_2 + (\beta_1 - \alpha_1 \frac{b_1}{a_1}) f_1 \]

\( \rightarrow \) modification of the transfer matrix.
Non-Uniform Meshes

Non-uniform meshes are a key component [Tavella & Randall 2000]

Define coordinate transformation

\[ Y = Y(\epsilon) \]

for \( n \) critical points \( B_k \) with density factors \( \beta_k \)

\[
\frac{dY(\epsilon)}{d\epsilon} = A \left[ \sum_{k=1}^{n} J_k(\epsilon)^{-2} \right]^{-\frac{1}{2}}
\]

\[
J_k(\epsilon) = \sqrt{\beta^2 + (Y(\epsilon) - B_k)^2}
\]

\[
Y(0) = Y_{min}
\]

\[
Y(1) = Y_{max}
\]

ODE solver is based on Peter’s Runge-Kutta implementation.

Example: \( x_0 = \ln(100), \nu_0 = 0.05 \), Feller constraint is fulfilled
Loss of Probability

Time evolution of the stationary distribution with zero flux condition.

\[ P(x) = \int_{-\infty}^{x} \hat{p}(\nu) d\nu \]
\[ \nu_{\text{min}} = P^{-1}(0.01) \]
\[ \nu_{\text{max}} = P^{-1}(0.99) \]
\[ \int_{\nu_{\text{min}}}^{\nu_{\text{max}}} \hat{p} d\nu = 0.98 \]

Integral error after evolving for one year:
\[ \left| \int_{\nu_{\text{min}}}^{\nu_{\text{max}}} p(\nu, t = 1y) d\nu - 0.98 \right| \]
Recap: Stationary distribution:

$$\hat{p}(\nu) = \beta^\alpha \nu^{\alpha-1} \exp(-\beta \nu) \Gamma(\alpha)^{-1}$$

Remove divergence following Lucic [2] by using

$$q = \nu^{1-\alpha} p$$

$$\Rightarrow \partial_t q = \frac{\sigma^2}{2} \nu \partial^2_{\nu} q + \kappa (\nu + \theta) \partial_{\nu} q + \frac{2\kappa^2 \theta}{\sigma^2} q$$

This equation has the stationary solution

$$\hat{q}(\nu) = \beta^\alpha \exp(-\beta \nu) \Gamma(\alpha)^{-1}$$

which converges to $\beta^\alpha \Gamma(\alpha)^{-1}$ as $\nu \to 0$
Time evolution of the transformed distribution with zero flux condition.
Apply Itô’s lemma to $z = \log \nu$:

$$dz = \left( (\kappa \theta - \frac{\sigma^2}{2}) \frac{1}{\nu} - \kappa \right) dt + \sigma \frac{1}{\sqrt{\nu}} dW$$

Fokker-Planck equation for the probability distribution $f : \mathbb{R} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, $(z, t) \mapsto f(z, t) \ (\nu = \exp(z))$:

$$\partial_t f(z, t) = -\partial_z \left( (\kappa \theta - \frac{\sigma^2}{2}) \frac{1}{\nu} - \kappa \right) f + \partial_z^2 \left( \frac{\sigma^2}{2} \frac{1}{\nu} f \right)$$

Stationary solution:

$$\hat{f}(z) = \beta^\alpha \exp(z \alpha) \exp(-\beta \exp(z)) \Gamma(\alpha)^{-1} = \nu \hat{p}(\nu)$$

$\hat{f}$ converges to 0 as $z \rightarrow -\infty$
Log Coordinates

Time evolution of log probability density with zero flux condition

\[ \log \text{FPE, grid size: } 100 \]
\[ \log \text{FPE, grid size: } 1000 \]

\[ \nu_{\min} = \min\left(0.001, F^{-1}(0.01)\right) \]
Proper implementation of the zero flux boundary condition is not enough to get a stable scheme.

Transformation of the PDE in log coordinates leads to a less poisonous problem.

Non-Uniform meshers are a key component for success.

→ all in all, mostly harmless 😊. Time for another dimension
Adding the stock process to the picture complicates matters a bit. Probability density has a second variable $x = \log S$, and the Fokker-Planck equation reads

$$
\partial_t f = \partial_z^2 A(z, x, t)f + \partial_z B(z, x, t)f + \partial_z \partial_x \rho C(z, x, t)f + \text{powers of } \partial_x
$$

Stretching the argument above a bit\(^1\) we arrive at the boundary condition

$$
|\partial_z A(z, x, t)f + B(z, x, t)f + \rho \partial_x C(z, x, t)f| \bigg|_{z=z_0,z_1} \equiv 0
$$

\(^1\)Can be made rigorous [2]
SLV Fokker-Planck: Natural Coordinates

\[
\begin{align*}
    dx_t &= (r_t - q_t - \frac{\nu_t}{2})dt + \sqrt{\nu_t}L(x, t)dW_t^x \\
    d\nu_t &= \kappa(\theta - \nu_t)dt + \eta\sigma \sqrt{\nu_t}dW_t^\nu \\
    \rho dt &= dW_t^x dW_t^\nu
\end{align*}
\]

Fokker-Planck equation:

\[
\begin{align*}
    \partial_t p &= \frac{1}{2} \partial_x^2 \left[ L^2 \nu p \right] + \frac{1}{2} \eta^2 \sigma^2 \partial_\nu^2 \left[ \nu p \right] + \eta \sigma \rho \partial_x \partial_\nu \left[ L\nu p \right] \\
    &\quad - \partial_x \left[ \left( r - q - \frac{1}{2} L^2 \nu \right) p \right] - \partial_\nu \left[ \kappa (\theta - \nu) p \right]
\end{align*}
\]

The zero flux condition takes the form for all \( x \):

\[
\left. \left[ \frac{\sigma^2}{2} \nu \partial_\nu p + \left( \kappa (\nu - \theta) + \frac{\sigma^2}{2} \right) p + \rho \nu \sigma \partial_x Lp \right] \right|_{\nu=\nu_0, \nu=\nu_{n+1}} = 0
\]
Fokker-Planck equation for \( q = \nu^{1-\alpha} p \)

\[
\partial_t q = \frac{\nu}{2} \partial_x^2 L^2 q + (-r_t + q_t) \partial_x q + \partial_x \left( \frac{\nu}{2} L^2 + \rho \sigma \frac{2\kappa \theta}{\sigma^2} L \right) q \\
+ \frac{\sigma^2}{2} \nu \partial_{\nu}^2 q + \kappa (\nu + \theta) \partial_{\nu} q + \frac{2\kappa^2 \theta}{\sigma^2} q \\
+ \rho \sigma \nu \partial_x \partial_{\nu} L q
\]

The zero flux condition takes the form \( \forall x : \)

\[
\left[ \frac{\sigma^2}{2} \nu \partial_{\nu} q + \kappa \nu q + \rho \nu \sigma \partial_x L q \right] \bigg|_{\nu=\nu_0, \nu=\nu_{n+1}} = 0
\]
SLV Fokker-Planck: Log Coordinates

\[
\begin{align*}
\, dx_t & = \left( r_t - q_t - \frac{\nu_t}{2} \right) dt + \sqrt{\nu_t} L(x, t) dW^x_t \\
\, dz_t & = \left( (\kappa \theta - \frac{\sigma^2}{2}) \frac{1}{\nu} - \kappa \right) dt + \eta \sigma \frac{1}{\sqrt{\nu}} dW^\nu_t \\
\, \rho dt & = dW^x_t dW^\nu_t
\end{align*}
\]

Fokker-Planck equation:

\[
\partial_t f = \frac{1}{2} \partial^2_x \left[ L^2 \nu f \right] + \frac{\eta^2 \sigma^2}{2} \partial^2_z \left[ \frac{1}{\nu} f \right] + \eta \sigma \rho \partial_x \partial_z \left[ L f \right] - \partial_x \left[ \left( r - q - \frac{1}{2} L^2 \nu \right) f \right] - \partial_z \left[ \left( (\kappa \theta - \frac{\sigma^2}{2}) \frac{1}{\nu} - \kappa \right) f \right]
\]

The zero-flux boundary condition is

\[
\left. \left[ \frac{\eta^2 \sigma^2}{2} \frac{1}{\nu} \partial_z f - \kappa \left( 1 - \frac{\theta}{\nu} \right) f + \rho \sigma \partial_x L f \right] \right|_{\nu = \nu_0, \nu = \nu_{n+1}} = 0
\]
Example log coordinates:

\[ \partial_t f = \frac{1}{2} \partial_x^2 \left[ L^2 \nu f \right] + \frac{\eta^2 \sigma^2}{2} \partial_z^2 \left[ \frac{1}{\nu} f \right] + \eta \sigma \rho \partial_x \partial_z \left[ Lf \right] \]

\[ - \partial_x \left[ \left( r - q - \frac{1}{2} L^2 \nu \right) f \right] - \partial_z \left[ \left( (\kappa \theta - \frac{\sigma^2}{2}) \frac{1}{\nu} - \kappa \right) f \right] \]

\[ \partial_t f = \frac{\nu}{2} \partial_x^2 L^2 f + \frac{\eta^2 \sigma^2}{2} \frac{1}{\nu} \partial_z^2 f + \eta \sigma \rho \partial_x \partial_z Lf \]

\[ + \left( -r + q \right) \partial_x f + \frac{\nu}{2} \partial_x L^2 f + \left[ \left( -\kappa \theta - \frac{\sigma^2}{2} \right) \frac{1}{\nu} + \kappa \right] \partial_z f + \frac{\kappa \theta}{\nu} f \]

Use multiplication of derivative operators with \( L \) on the right hand side, added method multR to \texttt{TripleBandBinearOp} (saves some terms).
Start Condition: Dirac Delta Distribution

To begin with the Dirac delta distribution need to be regularized. Approximation for small $\Delta t$ based on

$$L(x, t) = \frac{\sigma_{LV}(x_{t=0}, 0)}{\sqrt{\nu_0}} = \text{const} \forall t \in [0, \Delta t]$$

1. Exact solution is known for $\rho = 0$
2. One Euler Step based on the SDE leads to bivariate Gaussian distribution
3. Semi-Analytical solution for exact sampling [Brodie, Kaya 2006]
Start with a calibrated Local Volatility Model $\sigma_{LV}(x_t, t)$ and calibrated Heston Model $(\nu_0, \theta, \kappa, \sigma, \rho)$

Recap: Leverage $L(x_t, t)$ is given by

$$L(x_t, t) = \frac{\sigma_{LV}(x_t, t)}{\sqrt{\mathbb{E}[\nu_t \mid x = x_t]}} = \sigma_{LV}(x_t, t) \sqrt{\frac{\int_{\mathbb{R}^+} p(x_t, \nu, t) d\nu}{\int_{\mathbb{R}^+} \nu p(x_t, \nu, t) d\nu}}$$

Start condition: $p(x, \nu, 0) = \delta(x - x_0) \delta(\nu - nu_0)$

$$\Rightarrow L(x_{t=0}, 0) = \frac{\sigma_{LV}(x_{t=0}, 0)}{\sqrt{\nu_0}}$$
Iterative Scheme:

1. Use Fokker-Planck equation to get from

\[ p(x, \nu, t) \to p(x, \nu, t + \Delta t) \]

assuming a piecewise constant leverage function \( L(x_t, t) \) in \( t \)

2. Calculate leverage function at \( t + \Delta t \):

\[ L(x, t + \Delta t) = \sigma_{LV}(x, t + \Delta t) \sqrt{\frac{\int_{\mathbb{R}^+} p(x, \nu, t + \Delta t) d\nu}{\int_{\mathbb{R}^+} \nu p(x, \nu, t + \Delta t) d\nu}} \]

3. Set \( t := t + \Delta t \)

4. If \( t \) is smaller than the final maturity goto 1
Motivation: Set-up extreme test case for the LSV calibration

- Feller condition is strongly violated with $\alpha = 0.6$
- Implied volatility surface of the Heston and the local volatility model differ significantly.
- Local Volatility: $\sigma_{LV}(x, t) \equiv 30\%$
- Heston Parameters:
  $S_0 = 100, \sqrt{\nu_0} = 24.5\%, \kappa = 1, \theta = \nu_0, \sigma^2 = 0.2, \rho = -75\%$
- Use log coordinates and modified Craig-Sneyd scheme
Calibration Example: Heston Implied Volatility Surface
Calibration Example: Round Trip

Quality of calibration is tested by the round trip error

- Fokker-Planck step: Calibrate the leverage function $L(x, t)$
- Feyman-Kac step: Calculate European option prices under resulting LSV model and back out implied volatility surface
- Show differences w.r.t. expected value of

$$\sigma_{impl}(K, t) = \sigma_{LV}(S, t) = 30\%$$
Calibration Example: LSV Implied Volatility Surface
Calibration Example: Leverage Function $L(S_t, t)$

![Graph depicting $L(S_t, t)$ over time and underlying values]
Conclusion: Heston Local Volatility in QuantLib

- Backward Feynman-Kac solver
- Forward Fokker-Planck solver
  - Zero-Flux boundary condition
  - Natural and log coordinates, transformed probability density
- Non-uniform meshers are a key factor for success
- Heston Local Volatility calibration
- Round trip errors are around 5bp in vols for extreme case

Repository:
https://github.com/jschnetm/quantlib/tree/slv/QuantLib
William Feller. 
Two singular diffusion problems. 

Vladimir Lucic. 
Boundary conditions for computing densities in hybrid models via PDE methods. 

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