Choosing the Right Spread
Consistent Modelling of Funding and Tenor Basis

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1. Refresher Multi-Curve Pricing
   › Tenor- and Funding-Specific Yield Curves
   › Why Do We Need to Model the Basis?

2. Modelling Deterministic Tenor and Funding Basis
   › Continuous Compounded Funding Spreads
   › Simple and Continuous Compounded Tenor Spreads

3. Consistent Payoff-Adjustments for Multiple Funding Curves
   › Why Not Just Substitute Discount Curves?
   › What Can Go Wrong with Simple Compounded Spreads?

4. Deterministic Tenor and Funding Basis in QuantLib
   › Where Is the “Best” Place to Model the Basis?
   › Instruments, Models or Pricing Engines

5. Summary and References
Refresher Multi-Curve Pricing

› Tenor- and Funding-Specific Yield Curves
› Why Do We Need to Model the Basis?
Pre-Crisis Yield Curve Modelling

Single-Curve Setting

» Discount Factors

\[ P(t, T) = e^{-\int_t^T f(t, s) ds} = e^{-z(T)T} \]

» Forward Libor Rates

\[ L(t, T', T) = \left( \frac{P(t, T')}{P(t, T)} - 1 \right) \frac{1}{\Delta} \]

» (Discounted) Libor Coupons

\[ L(t, T', T) \cdot \Delta \cdot P(t, T') = P(t, T') - P(t, T) \]

» Derivative payoffs are expressed in terms of single interest rate curve

» Term structure models describe single interest rate curve dynamics, e.g. in terms of

  › Continuous compounded forward rate \[ f(t, T'), \]
  › Short rate \[ r(t) = f(t, t), \] or
  › Simple compounded (Libor) forward rate \[ L(t, T', T) \]
Differentiating Forwarding and Discounting Curves

Incorporate Tenor Forwarding Curve (e.g. 3M/6M Libor Curves)

- (Pseudo) Discount Factors $P^\Delta(t, T')$
- Forward Libor Rates
  \[
  L^\Delta(t, T', T) = \left[ \frac{P^\Delta(t, T')}{P^\Delta(t, T)} - 1 \right] \frac{1}{\Delta}
  \]
- (Discounted) Libor Coupons
  \[
  L^\Delta(t, T', T) \cdot \Delta \cdot P(t, T)
  \]

- Derivative payoffs are expressed in terms of two interest rate curves
  - Discount Curve
  - Tenor-specific Forwarding Curve

- Often some *deterministic spread* assumption is applied to allow using available models
Differentiating Funding Curves

Incorporate Funding-specific Discounting Curve (e.g. Cross-Currency Funding)

- Forward Libor Rates $L^A(t, T', T)$
- Cash Collateralizes Discounting with $P^{OIS}(t, T')$
- Cross-Currency Collateralizes Discounting with $P^{XCY}(t, T')$

Use Case:
- Calibrate model to cash collateralized (Eonia/OIS discounting) swaptions based on 6M Euribor Forwards
- Use model to price a USD cash collateralized (XCY discounting) derivative

Discounting spread could also originate from uncollateralised discounting or credit spread
Why Do We Need to Model the Basis?

Linear Products (e.g. Swaps)
- Only need \( L^\Delta(t, T', T) \), \( P^{OIS}(t, T) \), \( P^{XCY}(t, T) \)
- Require multi-curve bootstrapping
- Relation between curves irrelevant

Classical Term Structure Models
- Describe dynamics of only one curve
- Payoffs of Exotics may depend on various curves

Relation between curves required to evaluate exotic rate option payoffs
Tenor and Funding Basis Spreads

Multi-curve modelling via backbone curve plus basis spreads
Our Focus: Relation Between Tenor and Funding Basis Spreads

Multi-curve modelling requires consistent treatment of basis spreads
Modelling Deterministic Tenor and Funding Basis

› Continuous Compounded Funding Spreads
› Simple and Continuous Compounded Tenor Spreads
Continuous Compounded Funding Basis

Discount Factors

\[ P_{OIS}(t, T) = e^{-\int_t^T f_{OIS}(t,s) \, ds} \]

\[ P_{XCY}(t, T) = e^{-\int_t^T f_{XCY}(t,s) \, ds} \]

Continuous Compounded Funding Spread

\[ s(t, T) = f_{XCY}(t, T) - f_{OIS}(t, T) \]

Multiplicative Discount Factor Relation

\[ P_{XCY}(t, T) = P_{OIS}(t, T) \cdot D(t; t, T) \text{ with } \]

\[ D(t; t', T) = e^{-\int_t^{t'} s(t,s) \, ds} \]
Deterministic Funding Basis

Continuous Compounded Funding Spread

\[ s(t, T) = f^{XCY}(t, T) - f^{OIS}(t, T) \]

Assumption: \( s(t, T) \) is a deterministic function of \( t \) for all \( T \)

Forward Rate Modelling Relation

\[
d f^{XCY}(t, T) = d f^{OIS}(t, T) + \frac{\partial s(t, T)}{\partial t} dt
\]

» Equivalent forward rate volatility for OIS and XCY curve
» Equivalent volatilities of OIS and XCY zero coupon bonds \( P^{OIS}(t, T) \) and \( P^{XCY}(t, T) \)
» \( T \)-forward measure associated to numerairs \( P^{OIS}(t, T) \) and \( P^{XCY}(t, T) \) coincide, i.e.,

\[
E^{T,XCY}[.] = E^{T,OIS}[.]
\]

No convexity adjustment for switching between OIS and XCY discounting
Term Structure Models Using Deterministic Funding Basis

Forward Rate Modelling Relation

\[ df^{OIS}(t, T) = (\cdot)dt + \sigma(t, T)dW(t) \]

\[ df^{XCY}(t, T) = df^{OIS}(t, T) + \frac{\partial s(t, T)}{\partial t}dt \]

**Model Calibration**

» Model OIS curve \( f^{OIS}(t, T) \)

» Calibrate OIS curve based model parameters

» In particular vol structure \( \sigma(t, T) \)

**Derivative Pricing**

» Model XCY curve \( f^{XCY}(t, T) \)

» Use OIS curve based vol structure \( \sigma(t, T) \)

» Substitute \( f^{OIS} = f^{XCY} + s \)

Model parameters can be reused under deterministic funding basis assumption
Simple Compounded Tenor Basis

Forwarding Curve with $P^\Delta(t, T)$

Tenor basis spread

OIS Discounting Curve with $P^{OIS}(t, T)$

Simple Compounded Tenor Spread

$$B(t, T) = L^\Delta(t, T', T) - L^{OIS}(t, T', T)$$

Assumption: $B(t, T)$ is a deterministic function of $t$ for all $T$

Payoff adjustment at event date $t_e$

$$L^\Delta(t_e, T', T) = L^{OIS}(t_e, T', T) + B(t_e, T)$$

Forward Libor Rates

$$L^\Delta(t, T', T) = \left[ \frac{P^\Delta(t, T')}{P^\Delta(t, T)} - 1 \right] \frac{1}{\Delta}$$

OIS Forwards

$$L^{OIS}(t, T', T) = \left[ \frac{P^{OIS}(t, T')}{P^{OIS}(t, T)} - 1 \right] \frac{1}{\Delta}$$

Tenor basis results in static payoff adjustment, e.g. shift in strike for caplets
Simple Compounded Tenor Basis with XCY Discounting

Tenor Basis

\[ B(t, T) = L^\Delta(t, T', T) - L^{OIS}(t, T', T) \]

Funding Basis

\[ P^{XCY}(t, T) = P^{OIS}(t, T) \cdot D(t; t, T) \]

Payoff adjustment at event date \( t_e \)

\[ L^\Delta(t_e, T', T) = L^{OIS}(t_e, T', T) + B(t_e, T) \]

\[ = D(t_e, T', T) \cdot L^{XCY}(t_e, T', T) + B(t_e, T) + \frac{D(t_e, T', T) - 1}{\Delta} \]

Both tenor and funding basis required for static payoff adjustment
Continuous Compounded Tenor Basis

Continuous Compounded Tenor Forward Rates
\[
f^\Delta(t, T) = -\frac{\partial}{\partial T} \left[ \ln P^\Delta(t, T) \right]
\]

Continuous Compounded Tenor Spread
\[
b(t, T) = f^\Delta(t, T) - f^{OIS}(t, T)
\]
Assumption: \(b(t, T)\) is a deterministic function of \(t\) for all \(T\)

Multiplicative Discount Factor Relation
\[
P^\Delta(t, T) = P^{OIS}(t, T) \cdot D_b(t, t, T)^{-1} \text{ with } D_b(t, t, T) = e^{\int_T^t b(t, u)du}
\]

Payoff adjustment at event date \(t_e\)
\[
L^\Delta(t_e, T', T) = D_b(t_e, T', T) \cdot L^{OIS}(t_e, T', T) + \frac{D_b(t_e, T', T) - 1}{\Delta}
\]

Payoff adjustment results in affine transformation of OIS forwards
Continuous Compounded Tenor Basis with XCY Discounting

Tenor Basis

\[ P^\Delta(t, T) = P^{OIS}(t, T) \cdot D_b(t, t, T)^{-1} \]

Funding Basis

\[ P^{XCY}(t, T) = P^{OIS}(t, T) \cdot D(t; t, T) \]

Payoff adjustment at event date \( t_e \)

\[
L^\Delta(t_e, T', T) = D_b(t_e, T', T) \cdot L^{OIS}(t_e, T', T) + \frac{D_b(t_e, T', T) - 1}{\Delta} \\
= D_b(t_e, T', T) \cdot D(t_e, T', T) \cdot L^{XCY}(t_e, T', T) + \frac{D_b(t_e, T', T) \cdot D(t_e, T', T) - 1}{\Delta}
\]

Affine transformation payoff adjustment structure is preserved.
Continuous Compounded Tenor Basis with XCY Discounting (2)

Payoff adjustment at event date $t_e$

$$L^\Delta(t_e, T', T) = D_{b,XCY}(t_e, T', T) \cdot L^{XCY}(t_e, T', T) + \frac{D_{b,XCY}(t_e, T', T) - 1}{\Delta}$$

with

$$D_{b,XCY}(t_e, T', T) = D_b(t_e, T', T) \cdot D(t_e, T', T)$$

$$= e^\int_{T'}^T b(t,u) du \cdot e^{-\int_{T'}^T s(t,u) du}$$

$$= e^\int_{T'}^T [f^\Delta(t,u) - f^{OIS}(t,u)] du \cdot e^{-\int_{T'}^T [f^{XCY}(t,u) - f^{OIS}(t,u)] du}$$

$$= e^\int_{T'}^T [f^\Delta(t,u) - f^{XCY}(t,u)] du$$

Cont. Compounded Spread payoff adjustment is independent of OIS curve
## Summary Funding and Tenor Basis Payoff Adjustments

<table>
<thead>
<tr>
<th>Funding Basis</th>
<th>$p_{XCY} = p_{OIS} \cdot D$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Spread Convention</strong></td>
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Consistent Payoff-Adjustments for Multiple Funding Curves

› Why Not Just Substitute Discount Curves?
› What Can Go Wrong with Simple Compounded Spreads?
Modelling Funding Basis vs. Modelling Individual Discount Curves

Tenor and Funding Basis

Discounting and Tenor Basis

OIS discounting (e.g. calibration)

XCY discounting (e.g. pricing)
Consistently Modelling Individual Discount Curves

Consistent Model Dynamics

\[ df^{OIS} (t, T) = (\cdot) \cdot dt + \sigma^{OIS} \cdot dW(t) \]
\[ df^{XCY} (t, T) = df^{OIS} (t, T) + \frac{\partial s(t, T)}{\partial t} dt \]
\[ = (\cdot) \cdot dt + \sigma^{OIS} \sigma^{XCY} \cdot dW(t) \]

» Invariant Volatility structure (with deterministic shift)

\[ \sigma^{XCY} (f^{XCY} (t, T); t, T) = \sigma^{OIS} (f^{OIS} (t, T) - s(t, T); t, T) \]

» In particular unchanged short rate volatility and mean reversion for Hull White model

Uniform Payoff Adjustment Function

OIS discounting \( L^\Delta = G(L^{OIS}, f^{OIS}, f^\Delta) \)

XCY discounting \( L^\Delta = G(L^{XCY}, f^{XCY}, f^\Delta) \)

» Depends on spread compounding convention
Recall Funding and Tenor Basis Payoff Adjustments

| Funding Basis |  
|---------------|------------------------------------------------|
|               | \( P^XCY = P^{OIS} \cdot D \) |
| Spread Convention | Simple Compounded | Continuous Compounded |
| \( P^{OIS} \) | \( L^Δ = L^{OIS} + B \) | \( L^Δ = D_b \cdot L^{OIS} + \frac{D_b - 1}{Δ} \) |
| \( P^{XCY} \) | \( L^Δ = D \cdot L^{XCY} + B + \frac{D - 1}{Δ} \) | \( L^Δ = D_{b,XCY} \cdot L^{XCY} + \frac{D_{b,XCY} - 1}{Δ} \) |

Simple compounded tenor basis spreads do not yield uniform payoff adjustment function
## Enforcing Uniform Tenor Basis Payoff Adjustments

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<td><strong>Tenor Basis</strong></td>
<td>$p^{OIS}$</td>
</tr>
<tr>
<td>$p^{XCY}$</td>
<td>$L^\Delta = L^{XCY} + B^{XCY}$</td>
</tr>
</tbody>
</table>

Assume a deterministic basis $B^{XCY}$ in addition to deterministic terms $B$ and $D$.
Contradiction of Simultaneously Deterministic Terms $D$, $B$ and $B^{XCY}$

We have

$$L^{XCY}(t, T', T) - L^{OIS}(t, T', T) = \left[ \frac{P^{XCY}(t, T')}{P^{XCY}(t, T)} - \frac{P^{OIS}(t, T')}{P^{OIS}(t, T)} \right] \frac{1}{\Delta} = B(t, T) - B^{XCY}(t, T)$$

It follows

$$e^{\int_{T'}^{T} f^{XCY}(t,u)du} - e^{\int_{T'}^{T} f^{OIS}(t,u)du} = [B(t, T) - B^{XCY}(t, T)] \cdot \Delta$$

Solving for the funding spread yields

$$s(t, T) = f^{XCY}(t, T) - f^{OIS}(t, T) = \frac{\partial}{\partial T} \ln \left( 1 + \frac{[B(t, T) - B^{XCY}(t, T)] \cdot \Delta}{e^{\int_{T'}^{T} f^{OIS}(t,u)du}} \right)$$

Though $[B(t, T) - B^{XCY}(t, T)] \cdot \Delta$ deterministic, $s(t, T)$ depends on future forward rates $f^{OIS}(t, \cdot)$

Simple compounded tenor basis vs. XCY may only yield approximate payoff adjustment
Example: XCY Cash Collateralized Caplet with Simple Comp. Tenor Basis

We have

\[ Cpl^{OIS}(t) = P^{OIS}(t, T) E^{OIS}[(L(T', T', T) - k)^+ \cdot \Delta] \]

\[ Cpl^{XCY}(t) = P^{XCY}(t, T) E^{XCY}[(L(T', T', T) - k)^+ \cdot \Delta] \]

From \( E^{XCY}[\cdot] = E^{OIS}[\cdot] \) follows model-independent that

\[ Cpl^{XCY}(t) = D(t; t, T) \cdot Cpl^{OIS}(t) \]

Rewriting OIS caplet payoff as zero coupon bond put option and Hull White model

\[ Cpl^{OIS}(T') = (1 + [k - B(T)]\Delta) \cdot \left[ \frac{1}{1 + [k - B(T)]\Delta} - p^{OIS}(T', T) \right]^+ \]

\[ Cpl^{OIS}(t) = p^{OIS}(t, T') \cdot (1 + [k - B(T)]\Delta) \cdot B76 \left( \frac{p^{OIS}(t, T')}{p^{OIS}(t, T)} \cdot \frac{1}{1 + [k - B(T)]\Delta}, \sigma_P, -1 \right) \]
Correct vs. Simplified Payoff Adjustment XCY Cash Collateralized Caplet

We have

**Correct**

\[
C_{pl}^{XCY}(T') = (1 + [k - B(T)]\Delta) \cdot \left[ \frac{D(T', T)}{1 + [k - B(T)]\Delta} - P_{XCY}(T', T) \right]^{+}
\]

\[
\text{Variance Notional} \quad \text{Variance Strike}
\]

**Simplified**

\[
\overline{C_{pl}}^{XCY}(T') = (1 + [k - B^{XCY}(T)]\Delta) \cdot \left[ \frac{1}{1 + [k - B^{XCY}(T)]\Delta} - P_{XCY}(T', T) \right]^{+}
\]

Relative Valuation Error

\[
err_{Ntl} = \frac{1 + [k - B^{XCY}(T)]\Delta}{1 + [k - B(T)]\Delta} - 1 \approx (L^{XCY} - L^{OIS})\Delta
\]

\[
err_{B76} = \frac{B76 \left( \frac{P_{XCY}(t, T')}{P_{XCY}(t, T)}, \frac{D(T', T)}{1 + [k - B(T)]\Delta}, \sigma_p, -1 \right)}{B76 \left( \frac{P_{XCY}(t, T')}{P_{XCY}(t, T)}, \frac{D(T', T)}{1 + [k - B(T)]\Delta}, \sigma_p, -1 \right)} - 1
\]

\[
\approx \frac{\Phi \left( \frac{\sigma_p}{2} \right)}{2\Phi \left( \frac{\sigma_p}{2} \right) - 1} \cdot \frac{(L^{XCY} - L^{OIS})\Delta}{(1 + \Delta L^{XCY})(1 + \Delta L^{OIS})} \cdot (k - L^\Delta)
\]
Numerical Example XCY Cash Collateralized Caplet

» Yield curves for OIS/XCY/6M Forward flat at 100BP/50BP/200BP respectively
» Black caplet volatility 65%

1Y Caplets

Relative Error

Caplet Strike (BP)
Summary Consistent Payoff Adjustment

Continuous Compounded Tenor Basis

» Uniform payoff adjustment formula for OIS and XCY discounting

» Consistent multi-curve pricing with Hull White model by substituting discount curve

Simple Compounded Tenor Basis

» Different payoff adjustment formula for OIS and XCY discounting

» Approximations in multi-curve pricing with Hull White model by substituting discount curve

» Valuation error depends on cross currency basis and strike
Deterministic Tenor and Funding Basis in QuantLib

› Where Is the “Best” Place to Model the Basis?
› Transforming Instruments
› Generalising Models vs. Pricing Engines
Where Is the “Best” Place to Model the Basis?
Generalising Models

- Keep modelling assumptions and details in one place
- Manage `forwTermStructure_` consistent to `EuropeanSwaption→Index→TermStructure`
Generalising Pricing Engines

» By design knows forwarding and discounting term structure (no redundant information)

» Holding modelling assumptions out of the pricing model mixed up with instrument data
Disentangle spread modelling from existing yield curve modelling and pricing

Requires consistency of discounting curves between BondOption construction and HullWhiteModel

Model simple and continuous compounded Basis
Summary and Reference
Summary and Reference

**Modelling Deterministic Tenor and Funding Basis**
- Interdependencies of tenor and funding spreads
- Payoff adjustments for simple and continuous compounded tenor spreads
- Consistency for payoff adjustments with multiple funding curves

  *Continuous compounded spreads appear more favourable*

**Integrating Tenor and Funding Basis into QuantLib**
- Modifying Instrument, PricingEngine or Model classes

  *Best practice has yet to emerge*

**Further Reading**

S. Schlenkrich, A. Miemiec. *Choosing the Right Spread*. SSRN Preprint
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