UNCERTAIN VOLATILITY MODEL
Solving the Black Scholes Barenblatt Equation with the method of lines

GRM
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In 1973 Black, Scholes and Merton published their Option Pricing Model which later earned them the Nobel prize.

Following this paper it was now possible to price options without the need to know the real world drift of the underlying asset. They were constructing a risk less portfolio that growth at the risk free rate.

The problem which remained was the estimation of the volatility which is unknown and definitely not a constant.

Ever since then people have concentrated to model the remaining parameter “Volatility”.

However in 1995 Avellanada proposed a radical different approach by accepting the fact the volatility is unknown. In my talk I follow this path of Avellaneda and approach this problem from a sell side perspective using the uncertain volatility model.

It calculates a conservative price for an option portfolio given the option writer is willing to take on some risk in the black scholes delta hedging strategy.
The content of this presentation reflects the personal view and opinion of the author only. It does not express the view or opinion of HSH Nordbank AG on any subject presented in the following.
1. A simple Portfolio
2. Analysing the put spread
3. Motivation of the BSB Equation and the UV Model
4. Solving the BSB Equation with the method of lines
5. Results on our portfolio
6. Perspective
7. Numerical Details
8. References
A simple portfolio

Let's consider a portfolio consisting of 2 plain vanilla put options:

1. A long position in a put option struck at 110 EUR.
2. A short position in a put option struck at 90 EUR.

Both Options expire in 2 Years from now. The underlying of both options is a stock where we assume the following.

1) The riskless rate is at 0% and also the stock does not pay any dividends.
2) We assume today's Spot Price of the Underlying being at 90 EUR.
Payoff Profile of our Portfolio

![110-90 Put Spread](image)

- **Final Option Payoff**

**Spot**

- 0
- 5
- 10
- 15
- 20
- 25
- 30
- 35
- 40
- 45
- 50
- 55
- 60
- 65
- 70
- 75
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- 285
- 290
- 295
- 300

**Final Option Payoff**

- 20
- 15
- 10
- 5
- 0
Sensitivity to Volatility (Black Model)

- We took the QL Black Model.
- Then pricing the put spread for a range of volatilities

Diagram:
- Analysing the put-spread
- Vol
- Price

Volatility range: 0.1 to 0.6
There is no implied Black Volatility for this portfolio

Here the vega is 0
The underlying assumptions of the BSB Equation

\[ V^-(T,S) = F(S) \]

\[ \Gamma = \frac{\partial^2 V^-}{\partial S^2} \]

\[ 0 = \frac{\partial V^-}{\partial t} + \frac{1}{2} \sigma(\Gamma)^2 S^2 \frac{\partial^2 V^-}{\partial S^2} + rS \frac{\partial V^-}{\partial S} - rV^- \]

\[ \sigma(\Gamma) = \begin{cases} \sigma^+ & \text{if } \Gamma \geq 0 \\ \sigma^- & \text{if } \Gamma < 0 \end{cases} \]

\[ \sigma^- \leq \sigma \leq \sigma^+ \]

- In Wilmott (2006) is assumed to calculate a more conservative Price where a trader should sell the portfolio
- For calculating the best price (buy price) just interchange the roles of \( \sigma^+ \) and \( \sigma^- \) in the equation
- For a more rigorous mathematical derivation see Avellaneda (1995)
Key Portfolio Indicators

- Motivation of the BSB Equation

Diagram: Comparison Black vs UVM

- Black Price
- Worst Price Black Model
- Best Price Black Model
- Worst Price UVM
- Best Price UVM

Bid Ask Spread
- UVM
- Black Model
Numerical Solution of the BSB Equation with the method of lines

There are 2 methods of discretization:

- **Space** as shown in Hamdi (2007). This is the method we used here.
- **Time** as shown in Meyer (2003). We won't show this method in this presentation.

\[
0 = \frac{\partial V^-}{\partial t} + \frac{1}{2} \sigma (\Gamma)^2 S^2 \frac{\partial^2 V^-}{\partial S^2} + r S \frac{\partial V^-}{\partial S} - r V^- 
\]

The aim is to transform this PDE in a system of coupled ODEs which we then solve using the software in Ahnert (2011).

\[
\frac{dV^-}{dt} = f(t, V^-)
\]
Operator for ODEint

```cpp
struct bsb_operator
{
    bsb_operator(double r, double q, double h, double s_min, double sigma_low, double sigma_high, UInt price)
        : h(h), q(q), r(r), s_min(s_min), sigma_low(sigma_low), sigma_high(sigma_high), price(price)
    {
        // Keep the parameters of asset price and pde here
        double h, r, q, s_min, sigma_low, sigma_high;
        double s; // Holds AssetPrice
        UInt price; // Enumerator for Type of Price (WorstPrice or BestPrice)
        void operator()
            (const vector_type &x, vector_type &dxdt, double /* t */)
        {
            double sigma;
            int N_s = x.size() - 1;

            for (int l = 1; l < N_s - 1; l++)
            {
                double delta_t = (x[l + 1] - x[l - 1]) / (2*h); // FB Approximation for Delta
                double gamma = (x[l + 1] - 2*x[l] + x[l - 1]) / pow(h, 2); // Approximation for Gamma
                if (gamma < 0.0) // This is the difference to Black Scholes implements sigma(Gamma)
                {
                    if (price == UInt:WorstPrice)
                        sigma = sigma_low;
                    else
                        sigma = sigma_high;
                }
                else
                {
                    if (price == UInt:WorstPrice)
                        sigma = sigma_high;
                    else
                        sigma = sigma_low;
                }
                s = s_min + l*h;
                // Calculate the Option Theta using the pde
                dxdt[0] = -r * x[0];
                dxdt[l] = -r * x[l];
                //dxdt[0] = (q + r) * s_min * r * x[0];
                dxdt[1] = -r * x[1];
            }
    }
};
```

• Solving the BSB Equation with the method of lines
UVM Worst Price Surface

OptionPrices

Results on our portfolio
UVM Worst Price Gamma Surface

Results on our portfolio
UVM Worst Price Delta Surface

Results on our portfolio
Perspective

- The UVM PDE in non linear
- The present QuantLib offers pricing on an instrument level only and cannot handle portfolio effects as presented by the UVM (or CVA models)
- What could be further steps to raise the QL capacity to portfolio level?
Numerical Details and Hardware

- Grid has 600 Steps in Asset Price Direction.
- The grid spans from 1 EUR to 600 EUR in Asset Direction.
- The solution inside the odeint integration at time steps of 0.02 Years.
- The Bulirsch Stoer Stepper from ODEINT to integrate the ode system.
- It took about 1.5 seconds to solve for the price. With 300 Asset Steps it took 0.25 seconds
- The calculations were run on a pc running ubuntu.
- The processor has been an intel i7-7700K with 16 GB of RAM
The Odeint Library within boost

- We use our nice friend library boost. It ships with a nice numerical package named odeint.
- The authors of Odeint were Karsten Ahnert and Mario Mulansky.
- There is a lot of documentation to be found at http://www.odeint.com
References

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