Multi-Curve Convexity

CMS Pricing with Normal Volatilities and Basis Spreads in QuantLib

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Agenda

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2. Annuity Mapping Function as Conditional Expectation
3. Normal Model for CMS Coupons
4. Extending QuantLib’s CMS Pricing Framework
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CMS Payoff and Convexity Adjustment
CMS coupons refer to swap rates (like swaptions) but pay at a single pay date (unlike swaptions)

A forward swap rate is given as float leg over annuity

\[
S(t) = \frac{\sum_{i=1}^{N} L_i(t) \cdot \tau_i \cdot P(t, T_i)}{\sum_{j=1}^{M} \tau_j \cdot P(t, T_j)} \cdot \frac{A(t)}{A(t)}
\]

We consider a call on a (say 10y) swap rate \(S(T)\) fixed at \(T\),

\([S(T) - K]^+\) and paid at \(T_p \geq T\)

Payoff is evaluated under the annuity measure

\[
V(t) = A(t) \cdot E^{An} \left\{ \frac{P(t, T_p)}{An(T)} \cdot [S(T) - K]^+ \right\}
\]

› Annuity measure because swap rate dynamics are in principle available from swaption skew

› However, additional term \(P(T, T_p)/An(T)\) requires special treatment (convexity)

Tenor basis enters CMS pricing via swap rates (Libor forward curve) and additional discount terms (OIS discount curve)
CMS payoff may be decomposed into a Vanilla part and a remaining convexity adjustment part

Sometimes it makes sense to split up in Vanilla payoff and convexity adjustment

\[
V(t) = P(t, T_p) \cdot \left[ E^{An} \{[S(T) - K]^+] \right] + E^{An} \left\{ \left[ \frac{P(T, T_p)}{An(T)} \right] \frac{An(t)}{P(t, T_p)} - 1 \right\} \cdot (S(T) - K)^+
\]

Vanilla option          Convexity adjustment

What are the challenges for calculating the convexity adjustment?

» We know the dynamics of \( S(T) \) (under the annuity measure)

» We do not know the dynamics of \( P(T, T_p)/An(T) \)

» But it is reasonable to assume a very strong relation between \( S(T) \) and \( P(T, T_p)/An(T) \)

For CMS pricing we need to express \( P(T, T_p)/An(T) \) in terms of the swap rate \( S(T) \) taking into account tenor basis
Annuity Mapping Function as Conditional Expectation
Consider the iterated expectation

\[
E^{An} \left\{ E^{An} \left\{ \frac{P(T,T_p)}{An(T)} \cdot [S(T) - K]^+ \mid S(T) = s \right\} \right\} = E^{An} \left\{ E^{An} \left\{ \frac{P(T,T_p)}{An(T)} \mid S(T) = s \right\} \cdot [S(T) - K]^+ \right\}
\]

Define the **annuity mapping function**

\[
\alpha(s, T_p) = E^{An} \left\{ \frac{P(T,T_p)}{An(T)} \mid S(T) = s \right\}
\]

By construction \(\alpha(s, T_p)\) is deterministic in \(s\). We can write

\[
V(t) = An(t) \cdot E^{An} \{ \alpha(S(T), T_p) \cdot [S(T) - K]^+ \}
\]

\[
= An(t) \cdot \int_K^\infty \alpha(S(T), T_p) \cdot [S(T) - K]^+ \cdot dP(S(T))
\]

Conceptually, CMS pricing consists of **three steps**

1. Determine terminal distribution \(dP(S(T))\) of swap rate (in annuity measure)
2. Specify a model for the annuity mapping function \(\alpha(s, T_p)\)
3. Integrate payoff and annuity mapping function analytically (if possible) or numerically
In general tenor basis and multi-curve pricing affects CMS pricing by two means

\[ V(t) = An(t) \cdot \int_K^{\infty} \alpha(S(T), T_p) \cdot [S(T) - K]^+ \cdot dP(S(T)) \]

1. **Vanilla swaption pricing**
   - Required to determine terminal distribution
   - Use tenor forward curve and Eonia discount curve to calculate forward swap rate

2. **Construction of annuity mapping function**
   - Relate Eonia discount factors (and annuity) to swap rates based on tenor forward curve (and Eonia discount curve)

Multi-curve pricing for CMS coupons requires a basis model to specify the relation between discount factors and forward swap rate
How does an annuity mapping function look like in practice?

» For the Hull White model an annuity mapping function $\alpha(S(T), T_p)$ can easily be calculated

» This gives an impression of its functional form

The annuity mapping function shows less curvature in $S$- and $T$-direction. Thus it appears reasonable to apply linear approximations.
One modelling approach for the annuity mapping function is a linear terminal swap rate (TSR) model

Assume an affine functional relation for the annuity mapping function

\[ \alpha(s, T_p) = a(T_p) \cdot s + b(T_p) \]

for suitable time-dependent functions \( a(T_p) \) and \( b(T_p) \). By construction there is a fundamental no-arbitrage condition for TSR models

\[
E^{An}\{\alpha(S(T), T_p)\} = E^{An}\left\{E^{An}\left\{\frac{P(T, T_p)}{An(T)} | S(T) = s\right\}\right\} = E^{An}\left\{\frac{P(T, T_p)}{An(T)}\right\} = \frac{P(t, T_p)}{An(t)}
\]

From definition of the linear TSR model we get

\[
E^{An}\{\alpha(S(T), T_p)\} = E^{An}\{a(T_p) \cdot S(T) + b(T_p)\} = a(T_p) \cdot S(t) + b(T_p)
\]

Thus

\[
b(T_p) = \frac{P(t, T_p)}{An(t)} - a(T_p) \cdot S(t)
\]

This yields a linear TSR model representation only in terms of function \( a(T_p) \) as

\[
\alpha(s, T_p) = a(T_p) \cdot [s - S(t)] + \frac{P(t, T_p)}{An(t)}
\]

Linear TSR models only differ in their specification of the slope function \( a(T_p) \).

Slope function \( a(T_p) \) corresponds to \( G'(R_T^p) \) in Hagan’s Convexity Conundrums paper.
A further model-independent condition is given as additivity condition

Remember that $An(T) = \sum_{j=1}^{M} \tau_j \cdot P(T, \bar{T}_j)$. For all realisations $s$ of future swap rates $S(T)$ we have

$$\sum_{j=1}^{M} \tau_j \cdot \alpha(s, \bar{T}_j) = E^{An} \left\{ \sum_{j=1}^{M} \tau_j \cdot \frac{P(T, \bar{T}_j)}{An(T)} \mid S(T) = s \right\} = E^{An} \left\{ \frac{An(T)}{An(T)} \mid S(T) = s \right\} = 1$$

Applying the linear TSR model yields

$$\sum_{j=1}^{M} \tau_j \cdot \alpha(s, \bar{T}_j) = \left[ \sum_{j=1}^{M} \tau_j \cdot a(\bar{T}_j) \right] \cdot [s - S(t)] + \sum_{j=1}^{M} \tau_j \cdot \frac{P(t, T_j)}{An(t)} = 1$$

Thus **additivity condition** for slope function $a(\cdot)$ becomes

$$\left[ \sum_{j=1}^{M} \tau_j \cdot a(\bar{T}_j) \right] = 0$$

So far, no-arbitrage and additivity condition only depend on OIS discount factors. That is tenor basis does not affect them
Tenor basis is modelled as deterministic spread on continuous compounded forward rates for various tenors

6m Euribor tenor curve with forward rates \( L_i(t;T_{i-1},T_i) \)

Eonia/OIS discount curve with discount factor \( P(t,T) \)

Deterministic spread relation between forward rates

\[
f^{6m}(t,T) = f(t,T) + b(T)
\]

Deterministic Relation between forward Libor rates and OIS discount factors

\[
1 + \tau_i \cdot L_i(t) = D_i \cdot \frac{P(t,T_{i-1})}{P(t,T_i)} \text{ with } D_i = e^{\int_{T_{i-1}}^{T_i} b(s)ds}
\]

Swap rates may be expressed in terms of discount factors (without Libor rates)

\[
S(t) = \frac{\sum_{i=1}^{N} L_i(t) \cdot \tau_i \cdot P(t,T_i)}{\sum_{j=1}^{M} \tau_j \cdot P(t,T_j)} = \frac{\sum_{i=0}^{N} \omega_i \cdot P(t,T_i)}{\sum_{j=1}^{M} \tau_j \cdot P(t,T_j)} \text{ with } \omega_i = \begin{cases} D_1, & i = 0 \\ D_{i+1} - 1, & i = 1, \ldots, N - 1 \\ -1, & i = N \end{cases}
\]

We use the multiplicative terms \( D_i \) to describe tenor basis
Additional consistency condition links today’s forward swap rate to discount factors

We have for all realisations $s$ of future swap rates $S(T)$

$$
\sum_{i=0}^{N} \omega_i \cdot \alpha(s, T_i) = E^{An} \left\{ \frac{\sum_{i=0}^{N} \omega_i \cdot P(T, T_i)}{\sum_{j=1}^{M} \tau_j \cdot P(T, T_j)} \mid S(T) = s \right\} = E^{An} \{S(T) \mid S(T) = s\} = s
$$

Applying the linear TSR model yields

$$
\sum_{i=0}^{N} \omega_i \cdot \alpha(s, T_i) = \left[ \sum_{i=0}^{N} \omega_i \cdot a(T_i) \right] \cdot [s - S(t)] + \sum_{i=0}^{N} \omega_i \cdot \frac{P(t, T_i)}{An(t)} = s
$$

Above equations yield consistency condition specifying slope of $a(\cdot)$

$$
\sum_{i=0}^{N} \omega_i \cdot a(T_i) = 1
$$

Tenor basis enters coefficients $\omega_i$ (via spread terms $D_i$). Thus tenor basis has a slight effect of the slope of annuity mapping function in $T$-direction.
Additivity and consistency condition may be combined to fully specify an affine annuity mapping function

Necessary (additivity and consistency) conditions for a linear TSR model are

$$\sum_{j=1}^{M} \tau_j \cdot a(T_j) = 0 \quad \text{and} \quad \sum_{i=0}^{N} \omega_i \cdot a(T_i) = 1$$

If we set $a(T) = u \cdot (T_N - T) + v$ then we may directly solve for $u$ and $v$

$$u = - \frac{\sum_{j=1}^{M} \bar{\tau}_j}{[\sum_{j=1}^{M-1} \bar{\tau}_j (T_N - T_j)] \cdot [\sum_{i=0}^{N} \omega_i] - [\sum_{i=0}^{N-1} \omega_i (T_N - T_i)] \cdot [\sum_{j=1}^{M} \bar{\tau}_j]}$$

$$v = \frac{[\sum_{j=1}^{M-1} \bar{\tau}_j (T_N - T_j)]}{[\sum_{j=1}^{M-1} \bar{\tau}_j (T_N - T_j)] \cdot [\sum_{i=0}^{N} \omega_i] - [\sum_{i=0}^{N-1} \omega_i (T_N - T_i)] \cdot [\sum_{j=1}^{M} \bar{\tau}_j]}$$

There are more sophisticated approaches available to specify the annuity mapping function. However, to be fully consistent, they might need to be adapted to the consistency condition with tenor basis.
Comparing annuity mapping function in Hull White and affine TSR model shows reasonable approximation for relevant domain.

\[ \alpha(\cdot, T_p) \]

\[ \alpha(S(T), \cdot) \]
Normal Model for CMS Coupons
Applying linear TSR model to CMS instruments

\[ V(t) = P(t, T_p) \cdot \left[ E^A \{ [S(T) - K]^+] \right] + E^A \left\{ \frac{P(T, T_p)}{P(t, T_p)} \cdot \frac{An(t)}{An(T)} - 1 \right\} \cdot [S(T) - K]^+ \]

Replacing \( \frac{P(T, T_p)}{An(T)} \) by \( \alpha(s, T_p) = a(T_p) \cdot [S(T) - S(t)] + \frac{P(t, T_p)}{An(t)} \) (conditional expectation) yields

\[ CA(t) = E^A \left\{ \left[ a(T_p) \cdot [S(T) - S(t)] + \frac{P(t, T_p)}{An(t)} \cdot \frac{An(t)}{P(t, T_p)} - 1 \right] \cdot [S(T) - K]^+ \right\} \]

\[ = a(T_p) \cdot \frac{An(t)}{P(t, T_p)} \cdot E^A \{ [S(T) - S(t)] \cdot [S(T) - K]^+ \} \]

We do have a specification for slope function

How to solve for the expectation?

Solving for the expectation requires a model for the swap rate. Due to current low/negative interest rates we will apply a normal model.
Convexity adjustment for CMS calls consists of Vanilla option and power option

\[ [S(T) - S(t)] \cdot [S(T) - K]^+ = -[S(t) - K] \cdot [S(T) - K]^+ + 1_{\{S(T) \geq K\}} \cdot [S(T) - K]^2 \]

Convexity adjustment  
Vanilla option  
Power option

Vanilla option may be priced with Bachelier’s formula and implied normal volatility \( \sigma_N^K \)

Abbreviating \( v = \sigma_N^K \sqrt{T - t} \) and \( h = [S(t) - K]/v \) yields

\[ E^A_n\{[S(T) - K]^+\} = v[h \cdot N(h) + N'(h)] \]

Reusing the Vanilla model assumptions yields for the power option (after some algebra…)

\[ E^A_n\{1_{\{S(T) \geq K\}} \cdot [S(T) - K]^2\} = v^2[(h^2 + 1) \cdot N(h) + hN'(h)] \]

Convexity adjustment becomes

\[ E^A_n\{[S(T) - S(t)] \cdot [S(T) - K]^+\} = v^2 \cdot N(h) \]

Normal model yields compact formula for CMS convexity adjustment
Analogously we find normal model convexity adjustments for CMS floorlets and CMS swaplets \(^{(1)}\)

**CMS caplet**

\[
CA(t) = a(T_p) \cdot \frac{An(t)}{P(t, T_p)} \cdot v^2 \cdot N(h)
\]

**CMS floorlets**

\[
CA(t) = -a(T_p) \cdot \frac{An(t)}{P(t, T_p)} \cdot v^2 \cdot N(-h)
\]

**CMS swaplets**

\[
CA(t) = a(T_p) \cdot \frac{An(t)}{P(t, T_p)} \cdot v^2
\]

\(^{(1)}\) Normal model CMS convexity adjustment formulas are also stated in a preprint Version of Hagan 2003

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\(2016-07-12\) | Multi-Curve Convexity | Normal Model for CMS Coupons (3/5)
Example CMS convexity adjustments for June'16 market data based on Normal model

**Single Curve:**
- Calculate swap rate and convex. adjustment only by 6m Euribor curve

**Multi Curve:**
- Calculate swap rate and conv. adjustment by 6m Euribor forward and Eonia discount curve

**Affine:**
- Affine TSR model (with basis spreads)

**Standard:**
- (linearised) standard yield curve model (see Hag’03)

**Mean Rev. 10%:**
- (linearised) yield curve model based on mean reverting shifts (mean rev. 10%) (see Hag’03)
Model-implied 10y CMS swap spreads of Normal model show reasonable fit to quoted market data (1)

(1) Quotation 10y CMS swap spread: 10y CMS rate vs. 3m Euribor + quoted spread
Extending QuantLib‘s CMS Pricing Framework
There is a flexible framework for CMS pricing in QuantLib which can easily be extended.

We focus on the framework specified in the HaganPricer class.
We add analytic formulas for Normal dynamics and affine TSR model with basis spreads

CMS framework in QuantLib allows easy modifications and extensions, e.g., generalising NumericHaganPricer to normal or shifted log-normal volatilities.
Summary and References
Summary

» Current low interest rates market environment requires generalisation of classical log-normal based CMS convexity adjustment formulas

» Normal model for CMS pricing is easily be incorporated into QuantLib and yields good fit to CMS swap quotes

» Tenor basis impacts specification of TSR models – however modelling effect is limited compared to other factors

References


» https://github.com/sschlenkrich/quantlib-old
d-fine