Multi-Curve Convexity

CMS Pricing with Normal Volatilities and Basis Spreads in QuantLib

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Agenda

1. CMS Payoff and Convexity Adjustment
2. Annuity Mapping Function as Conditional Expectation
3. Normal Model for CMS Coupons
4. Extending QuantLib‘s CMS Pricing Framework
5. Summary and References

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CMS Payoff and Convexity Adjustment

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CMS coupons refer to swap rates (like swaptions) but pay at a single pay date (unlike swaptions)

A forward swap rate is given as float leg over annuity

$$S(t) = \frac{\sum_{i=1}^{N} L_i(t) \cdot \tau_i \cdot P(t, T_i)}{\underbrace{\sum_{j=1}^{M} \tau_j \cdot P(t, \bar{T}_j)}_{An(t)}}$$

We consider a call on a (say 10y) swap rate $S(T)$ fixed at T,

$$[S(T) - K]^+ \text{ and paid at } T_p \geq T$$

Payoff is evaluated under the annuity meassure

$$V(t) = An(t) \cdot E^{An}\left\{\frac{P(T, T_p)}{An(T)} \cdot [S(T) - K]^+\right\}$$

- Annuity meassure because swap rate dynamics are in principle available from swaption skew
- However, additional term $P(T, T_p)/An(T)$ requires special treatment (convexity)

Tenor basis enters CMS pricing via swap rates (Libor forward curve) and additional discount terms (OIS discount curve)

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CMS payoff may be decomposed into a Vanilla part and a remaining convexity adjustment part

Sometimes it makes sense to split up in Vanilla payoff and convexity adjustment

$$V(t) = P(t,T_p) \cdot \left[E^{An}\{[S(T)-K]^+\} + E^{An}\left\{ \left[\frac{P(T,T_p)}{An(T)} \frac{An(t)}{P(t,T_p)} - 1 \right] \cdot [S(T)-K]^+ \right\} \right]$$

Vanilla option Convexity adjustment

What are the challenges for calculating the convexity adjustment?

» We know the dynamics of $S(T)$ (under the annuity meassure)

» We do not know the dynamics of $P(T,T_p)/An(T)$

» But it is reasonable to assume a very strong relation between $S(T)$ and $P(T,T_p)/An(T)$

For CMS pricing we need to express $P(T,T_p)/An(T)$ in terms of the swap rate $S(T)$ taking into account tenor basis

Annuity Mapping Function as Conditional Expectation

Quotient $P(T,T_p)/An(T)$ is expressed in terms of the swap rate $S(T)$ by means of an annuity mapping function

Consider the iterated expectation

$$E^{An}\left\{E^{An}\left\{\frac{P(T,T_p)}{An(T)}\cdot[S(T)-K]^+ \mid S(T)=s\right\}\right\} = E^{An}\left\{E^{An}\left\{\frac{P(T,T_p)}{An(T)} \mid S(T)=s\right\}\cdot[S(T)-K]^+\right\}$$

Define the **annuity mapping function**

$$\alpha\left(s,T_p\right) = E^{An}\left\{\frac{P(T,T_p)}{An(T)} \mid S(T)=s\right\}$$

By construction $\alpha\left(s,T_p\right)$ is deterministic in s. We can write

$$V(t) = An(t)\cdot E^{An}\left\{\alpha\left(S(T),T_p\right)\cdot[S(T)-K]^+\right\}$$

$$= An(t)\cdot\int_K^\infty \alpha\left(S(T),T_p\right)\cdot[S(T)-K]^+\cdot dP\left(S(T)\right)$$

Conceptually, CMS pricing consists of **three steps**

1. Determine terminal distrubution $dP\left(S(T)\right)$ of swap rate (in annuity meassure)
2. Specify a model for the annuity mapping function $\alpha\left(s,T_p\right)$
3. Integrate payoff and annuity mapping function analytically (if possible) or numerically

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In general tenor basis and multi-curve pricing affects CMS pricing by two means

$$V(t) = An(t) \cdot \int_K^\infty \alpha\left(S(T), T_p\right) \cdot [S(T) - K]^+ \cdot dP\left(S(T)\right)$$

1. Vanilla swaption pricing

- Required to determine terminal distribution
- Use tenor forward curve and Eonia discount curve to calculate forward swap rate

2. Construction of annuity mapping function

- Relate Eonia discount factors (and annuity) to swap rates based on tenor forward curve (and Eonia discount curve)

> Multi-curve pricing for CMS coupons requires a basis model to specify the relation between discount factors and forward swap rate

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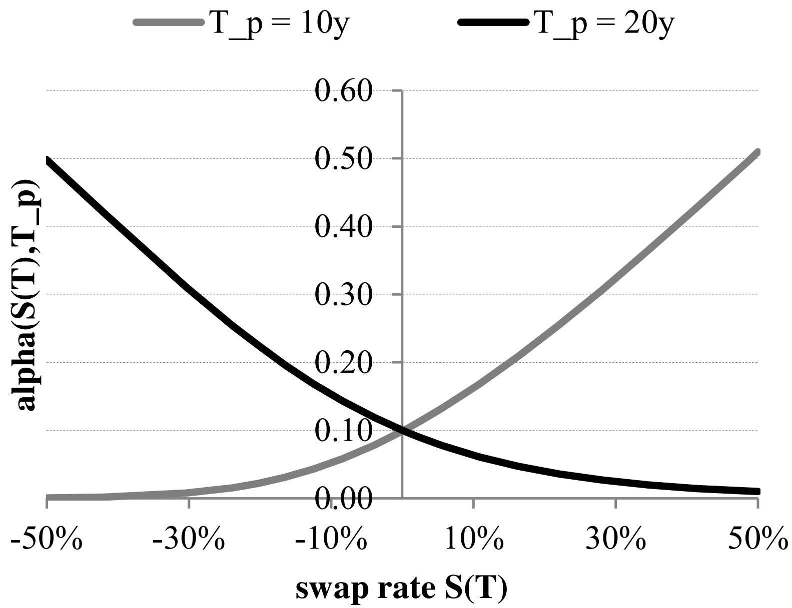
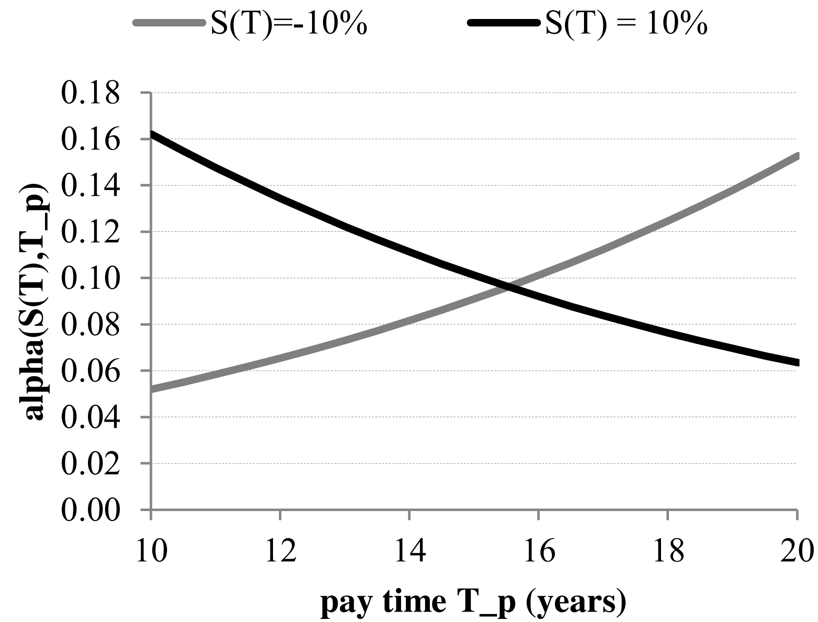
How does an annuity mapping function look like in practice?

- For the Hull White model an annuity mapping function $\alpha(S(T), T_p)$ can easily be calculated
- This gives an impression of its functional form

$\alpha(\cdot, T_p)$



$\alpha(S(T), \cdot)$



The annuity mapping function shows less curvature in S- and T-direction. Thus it appears reasonable to apply linear approximations

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One modelling approach for the annuity mapping function is a linear terminal swap rate (TSR) model

Assume an affine functional relation for the annuity mapping function

$$\alpha(s, T_p) = a(T_p) \cdot s + b(T_p)$$

for suitable time-dependent functions $a(T_p)$ and $b(T_p)$. By construction there is a fundamental **no-arbitrage condition** for TSR models

$$E^{An}\{\alpha(S(T), T_p)\} = E^{An}\left\{E^{An}\left\{\frac{P(T, T_p)}{An(T)} \mid S(T) = s\right\}\right\} = E^{An}\left\{\frac{P(T, T_p)}{An(T)}\right\} = \frac{P(t, T_p)}{An(t)}$$

From definition of the linear TSR model we get

$$E^{An}\{\alpha(S(T), T_p)\} = E^{An}\{a(T_p) \cdot S(T) + b(T_p)\} = a(T_p) \cdot S(t) + b(T_p)$$

Thus

$$b(T_p) = \frac{P(t, T_p)}{An(t)} - a(T_p) \cdot S(t)$$

This yields a linear TSR model representation only in terms of function $a(T_p)$ as(1)

$$\boldsymbol{\alpha(s, T_p) = a(T_p) \cdot [s - S(t)] + \frac{P(t, T_p)}{An(t)}}$$

Linear TSR models only differ in their specification of the slope function $a(T_p)$.

Slope function $a(T_p)$ corresponds to $G'(R_S^0)$ in Hagan's Convexity Conundrums paper

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A further model-independent condition is given as additivity condition

Remember that $An(T) = \sum_{j=1}^{M} \tau_j \cdot P(T, \bar{T}_j)$. For all realisations s of future swap rates $S(T)$ we have

$$\sum_{j=1}^{M} \tau_j \cdot \alpha(s, \bar{T}_j) = E^{An}\left\{\sum_{j=1}^{M} \tau_j \cdot \frac{P(T, \bar{T}_j)}{An(T)} \mid S(T) = s\right\} = E^{An}\left\{\frac{An(T)}{An(T)} \mid S(T) = s\right\} = 1$$

Applying the linear TSR model yields

$$\sum_{j=1}^{M} \tau_j \cdot \alpha(s, \bar{T}_j) = \left[\sum_{j=1}^{M} \tau_j \cdot a(\bar{T}_j)\right] \cdot [s - S(t)] + \underbrace{\sum_{j=1}^{M} \tau_j \cdot \frac{P(t, T_j)}{An(t)}}_{1} = 1$$

Thus **additivity condition** for slope function $a(\cdot)$ becomes

$$\left[\sum_{j=1}^{M} \tau_j \cdot a(\bar{T}_j)\right] = 0$$

So far, no-arbitrage and additivity condition only depend on OIS discount factors. That is tenor basis does not affect them

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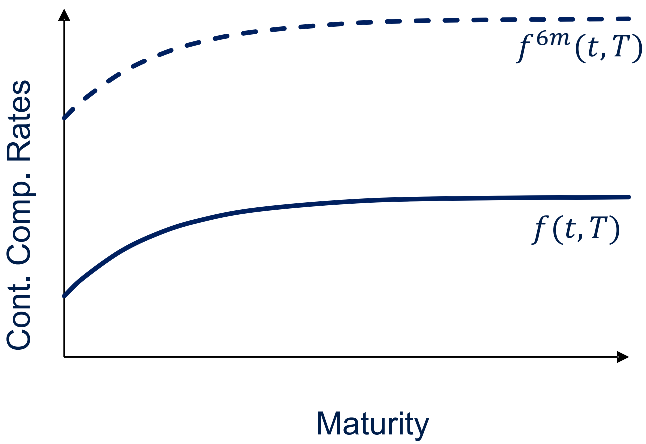
Tenor basis is modelled as deterministic spread on continuous compounded forward rates for various tenors



6m Euribor tenor curve with forward rates $L_i(t; T_{i-1}, T_i)$

Eonia/OIS discount curve with discount factor $P(t,T)$

Deterministic spread relation between forward rates

$$f^{6m}(t,T) = f(t,T) + b(T)$$

Deterministic Relation betwen forward Libor rates and OIS discount factors

$$1 + \tau_i \cdot L_i(t) = D_i \cdot \frac{P(t,T_{i-1})}{P(t,T_i)} \text{ with } D_i = e^{\int_{T_{i-1}}^{T_i} b(s)ds}$$

Swap rates may be expressed in terms of discount factors (without Libor rates)

$$S(t) = \frac{\sum_{i=1}^{N} L_i(t) \cdot \tau_i \cdot P(t,T_i)}{\sum_{j=1}^{M} \tau_j \cdot P(t,\bar{T}_j)} = \frac{\sum_{i=0}^{N} \omega_i \cdot P(t,T_i)}{\sum_{j=1}^{M} \tau_j \cdot P(t,\bar{T}_j)} \quad \text{with} \quad \omega_i = \begin{cases} D_1, & i = 0 \\ D_{i+1} - 1, & i = 1, \dots, N-1 \\ -1, & i = N \end{cases}$$

We use the multiplicative terms D_i to describe tenor basis

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Additional consistency condition links today‘s forward swap rate to discount factors

We have for all realisations s of future swap rates $S(T)$

$$\sum_{i=0}^{N} \omega_i \cdot \alpha(s, T_i) = E^{An}\left\{\frac{\sum_{i=0}^{N} \omega_i \cdot P(T, T_i)}{\sum_{j=1}^{M} \tau_j \cdot P(T, \bar{T}_j)} \mid S(T) = s\right\} = E^{An}\{S(T) | S(T) = s\} = s$$

Applying the linear TSR model yields

$$\sum_{i=0}^{N} \omega_i \cdot \alpha(s, T_i) = \left[\sum_{i=0}^{N} \omega_i \cdot a(T_i)\right] \cdot [s - S(t)] + \underbrace{\sum_{i=0}^{N} \omega_i \cdot \frac{P(t, T_i)}{An(t)}}_{S(t)} = s$$

Above equations yield consistency condition specifying slope of $a(\cdot)$

$$\sum_{i=0}^{N} \omega_i \cdot a(T_i) = 1$$

Tenor basis enters coefficients ω_i (via spread terms D_i). Thus tenor basis has a slight effect of the slope of annuity mapping function in T-direction

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Additivity and consistency condition may be combined to fully specify an affine annuity mapping function

Neccessary (additivity and consistency) conditions for a linear TSR model are

$$\sum_{j=1}^{M} \tau_j \cdot a(\bar{T}_j) = 0 \quad \text{and} \quad \sum_{i=0}^{N} \omega_i \cdot a(T_i) = 1$$

If we set $a(T) = u \cdot (T_N - T) + v$ then we may directly solve for u and v

$$u = -\frac{\sum_{j=1}^{M} \bar{\tau}_j}{\left[\sum_{j=1}^{M-1} \bar{\tau}_j \left(T_N - \bar{T}_j\right)\right] \cdot \left[\sum_{i=0}^{N} \omega_i\right] - \left[\sum_{i=0}^{N-1} \omega_i (T_N - T_i)\right] \cdot \left[\sum_{j=1}^{M} \bar{\tau}_j\right]}$$

$$v = \frac{\left[\sum_{j=1}^{M-1} \bar{\tau}_j \left(T_N - \bar{T}_j\right)\right]}{\left[\sum_{j=1}^{M-1} \bar{\tau}_j \left(T_N - \bar{T}_j\right)\right] \cdot \left[\sum_{i=0}^{N} \omega_i\right] - \left[\sum_{i=0}^{N-1} \omega_i (T_N - T_i)\right] \cdot \left[\sum_{j=1}^{M} \bar{\tau}_j\right]}$$

There are more sophisticated approaches available to specify the annuity mapping function. However, to be fully consistent, they might need to be adapted to the consistency condition with tenor basis

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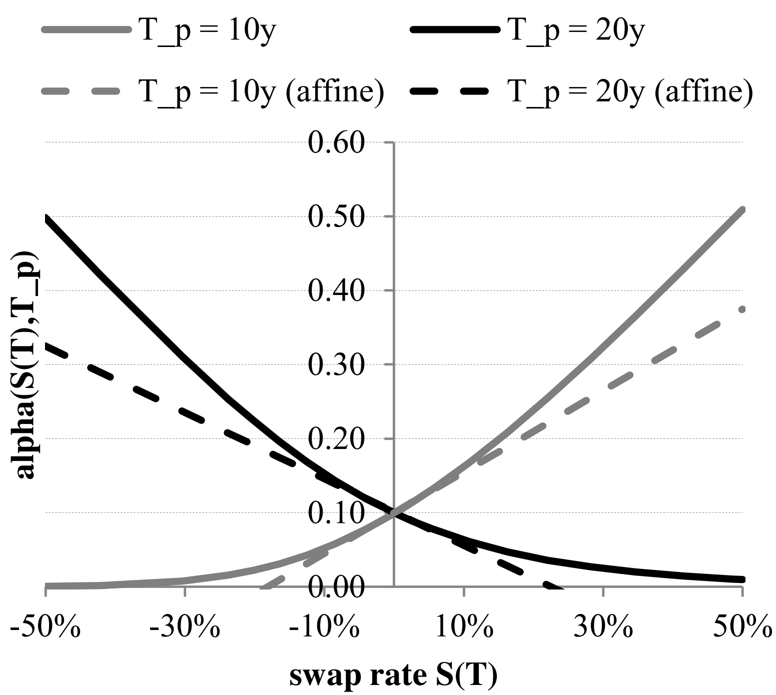
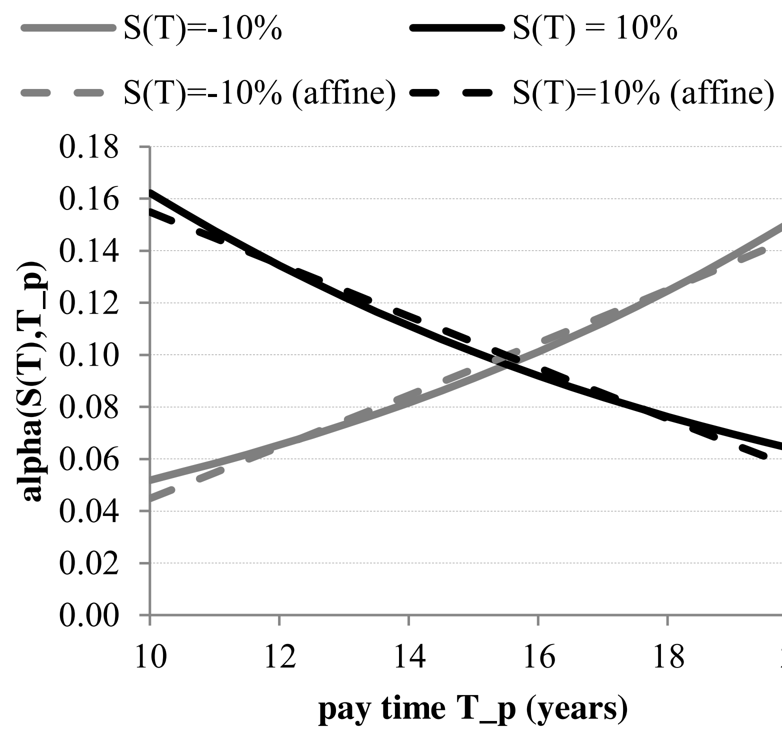
Comparing annuity mapping function in Hull White and affine TSR model shows reasonable approximation for relevant domain

$\alpha(\cdot, T_p)$



$\alpha(S(T), \cdot)$



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Normal Model for CMS Coupons

Applying linear TSR model to CMS instruments

$$V(t) = P(t,T_p) \cdot \left[E^{An}\{[S(T)-K]^+\} + \underbrace{E^{An}\left\{ \left[\frac{P(T,T_p)}{An(T)} \frac{An(t)}{P(t,T_p)} - 1 \right] \cdot [S(T)-K]^+ \right\}}_{CA(t)} \right]$$

Replacing $\frac{P(T,T_p)}{An(T)}$ by $\alpha(s,T_p) = a(T_p) \cdot [S(T) - S(t)] + \frac{P(t,T_p)}{An(t)}$ (conditional expectation) yields

$$CA(t) = E^{An}\left\{ \left[\left[a(T_p) \cdot [S(T)-S(t)] + \frac{P(t,T_p)}{An(t)} \right] \frac{An(t)}{P(t,T_p)} - 1 \right] \cdot [S(T)-K]^+ \right\}$$

$$= a(T_p) \cdot \frac{An(t)}{P(t,T_p)} \cdot E^{An}\{[S(T)-S(t)] \cdot [S(T)-K]^+\}$$

We do have a specification for slope function

How to solve for the expectation?

> Solving for the expectation requires a model for the swap rate. Due to current low/negative interest rates we will apply a normal model.

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Convexity adjustment for CMS calls consists of Vanilla option and power option

$$[S(T) - S(t)] \cdot [S(T) - K]^+ = -[S(t) - K] \cdot [S(T) - K]^+ + 1_{\{S(T) \geq K\}} \cdot [S(T) - K]^2$$

Convexity adjustment — Vanilla option — Power option

Vanilla option may be priced with Bachelier‘s formula and implied normal volatility σ_N^K

Abbreviating $\nu = \sigma_N^K \sqrt{T - t}$ and $h = [S(t) - K]/\nu$ yields

$$E^{An}\{[S(T) - K]^+\} = \nu[h \cdot N(h) + N'(h)]$$

Reusing the Vanilla model assumptions yields for the power option (after some algebra…)

$$E^{An}\left\{1_{\{S(T) \geq K\}} \cdot [S(T) - K]^2\right\} = \nu^2[(h^2 + 1) \cdot N(h) + hN'(h)]$$

Convexity adjustment becomes

$$E^{An}\{[S(T) - S(t)] \cdot [S(T) - K]^+\} = \nu^2 \cdot N(h)$$

Normal model yields compact formula for CMS convexity adjustment

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Analogously we find normal model convexity adjustments for CMS floorlets and CMS swaplets (1)

CMS caplet

$$CA(t) = a(T_p) \cdot \frac{An(t)}{P(t, T_p)} \cdot \nu^2 \cdot N(h)$$

CMS floorlets

$$CA(t) = -a(T_p) \cdot \frac{An(t)}{P(t, T_p)} \cdot \nu^2 \cdot N(-h)$$

CMS swaplets

$$CA(t) = a(T_p) \cdot \frac{An(t)}{P(t, T_p)} \cdot \nu^2$$

(1) Normal model CMS convexity adjustment formulas are also stated in a preprint Version of Hagan 2003

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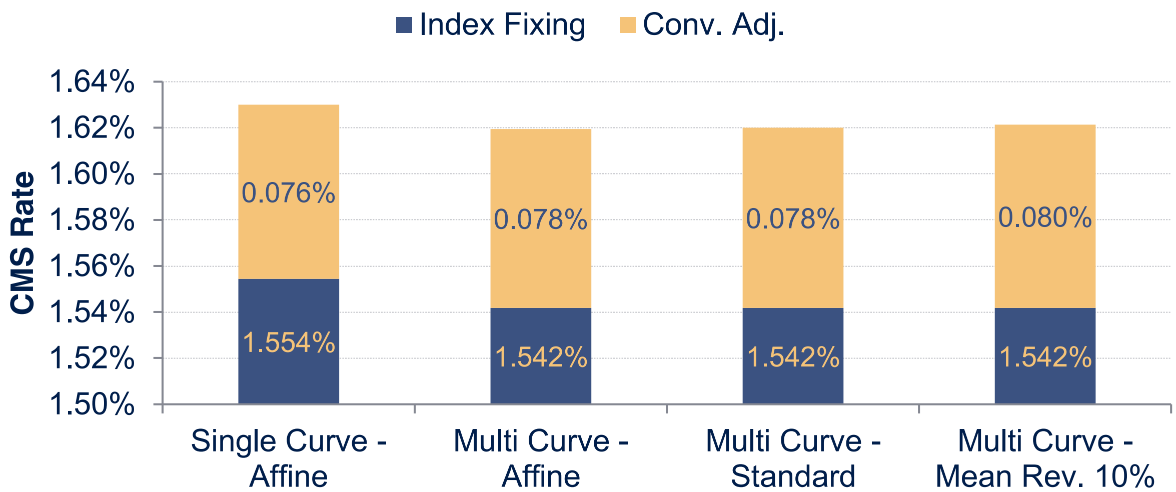
Example CMS convexity adjustments for June'16 market data based on Normal model

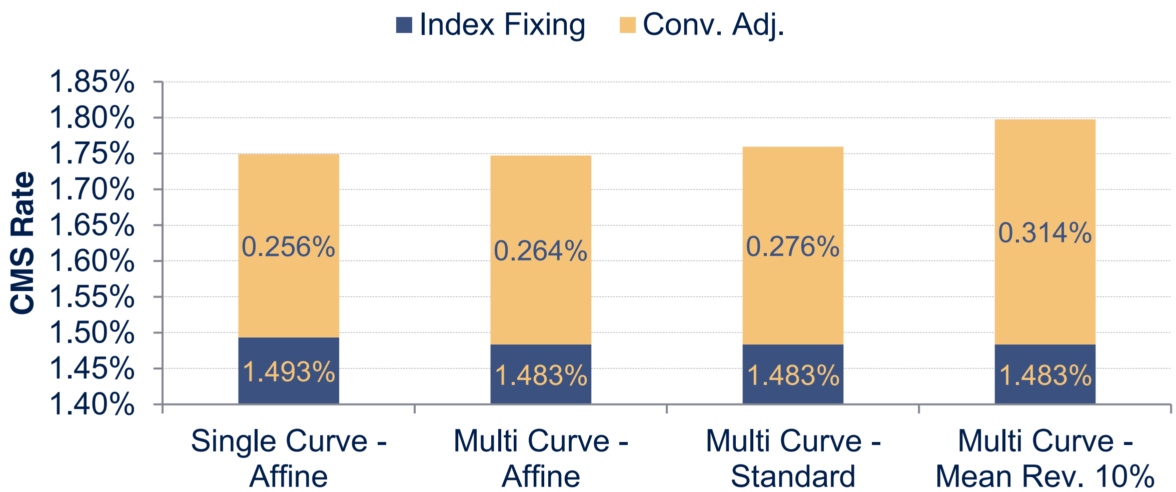
10y x 2y

CMS rate



10y x 10y

CMS rate



Single Curve:

» Calculate swaprate and conv. adjustment only by 6m Euribor curve

Multi Curve:

» Calculate swaprate and conv. adjustment by 6m Euribor forward and Eonia discount curve

Affine:

» Affine TSR model (with basis spreads)

Standard:

» (linearised) standard yield curve model (see Hag'03)

Mean Rev. 10%:

» (linearised) yield curve model based on mean reverting shifts (mean rev. 10%) (see Hag'03)

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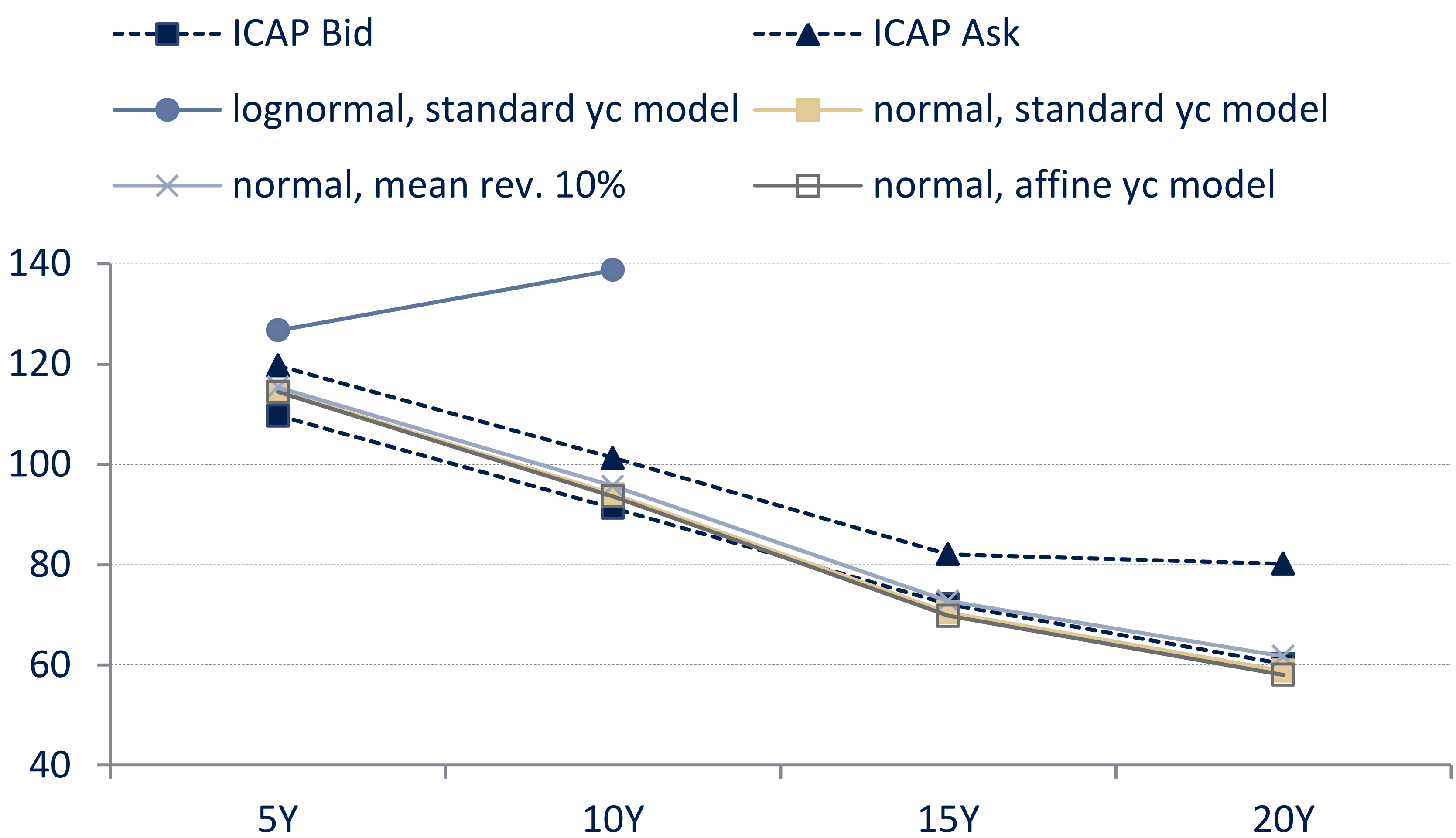
Model-implied 10y CMS swap spreads of Normal model show reasonable fit to quoted market data (1)



(1) Quotation 10y CMS swap spread: 10y CMS rate vs. 3m Euribor + quoted spread

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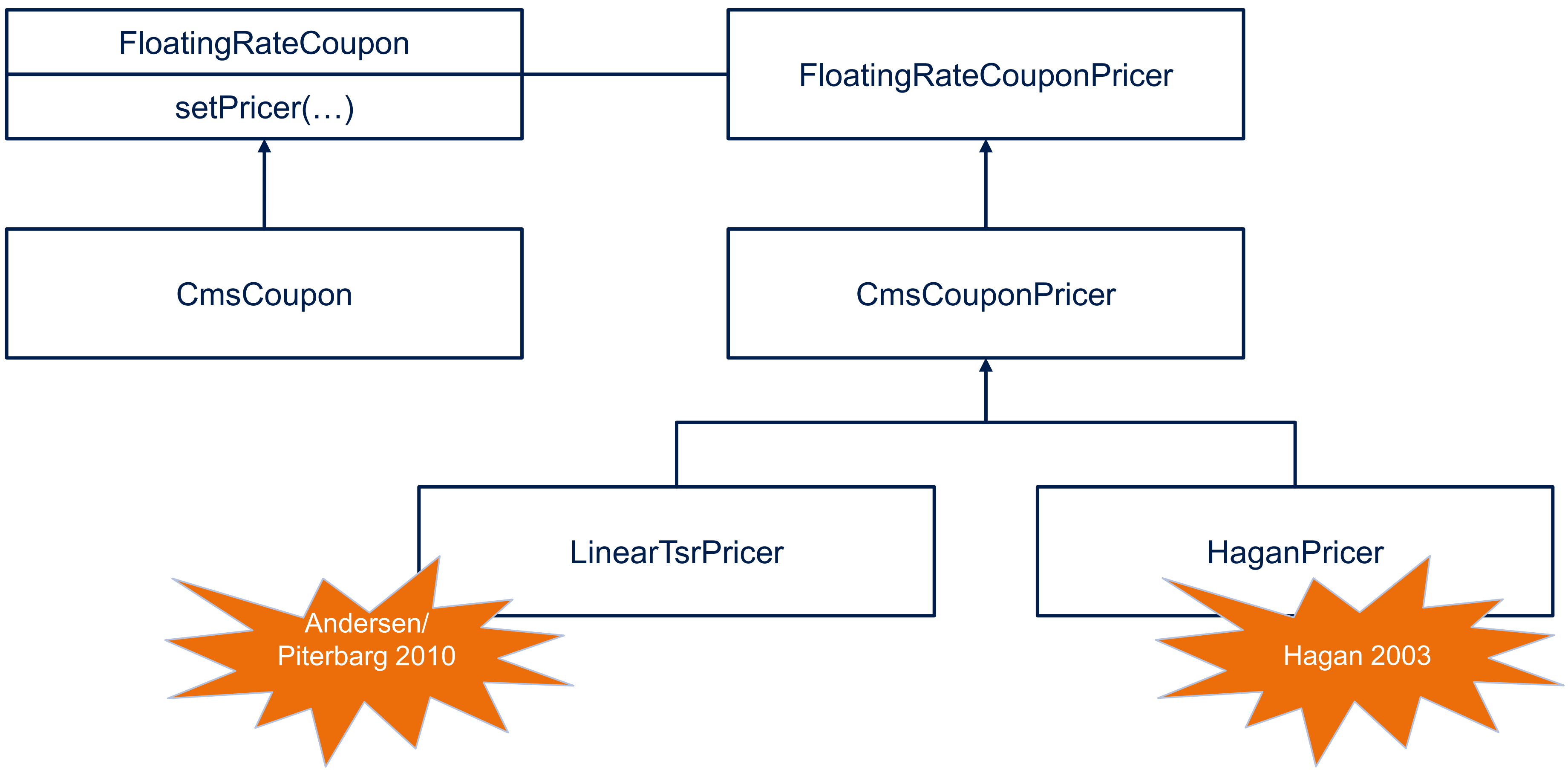
Extending QuantLib‘s CMS Pricing Framework

There is a flexible framework for CMS pricing in QuantLib which can easily be extended



We focus on the framework specified in the HaganPricer class

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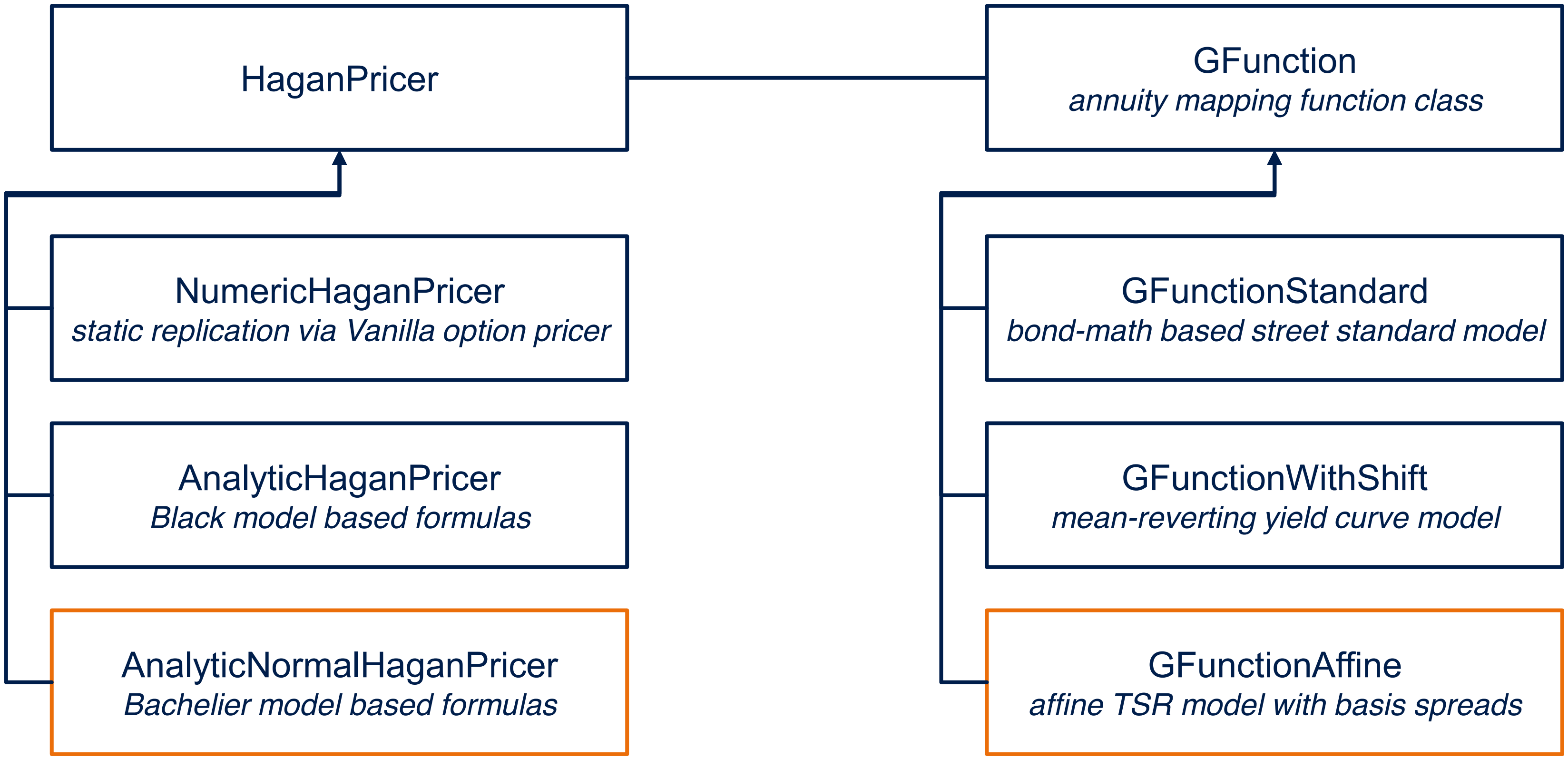
We add analytic formulas for Normal dynamics and affine TSR model with basis spreads



CMS framework in QuantLib allows easy modifications and extensions, e.g., generalising NumericHaganPricer to normal or shifted log-normal volatilities

Summary and References

Summary

» Current low interest rates market environment requires generalisation of classical log-normal based CMS convexity adjustment formulas

» Normal model for CMS pricing is easily be incorporated into QuantLib and yields good fit to CMS swap quotes

» Tenor basis impacts specification of TSR models – however modelling effect is limited compared to other factors

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» https://github.com/sschlenkrich/quantlib-old

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