Overview
Negative Rates, SABR PDE and Approximation

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Presenter

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DISCLAIMER

Negative Rates

- Markets, Models, Numerics -

The good old times - Markets

Interest rate World

- Rates are significantly positive
- Volatilities are at „normal“ levels
- Quotes are in log-normal volatility or premium
- There was a simple to code approach for SABR to a model which gives a very well fit of the market observed volatility structures

$$dS(t) = S\sigma dW(t), \quad S(0) = s_0$$

$$\begin{aligned} C_{LN}(S(0), K, T, \sigma) &= S(0)\mathcal{N}(d_1) - K\mathcal{N}(d_2) \\ P_{LN}(S(0), K, T, \sigma) &= K\mathcal{N}(-d_2) - S(0)\mathcal{N}(-d_1) \end{aligned}$$

SABR
SDE

The (popular) SABR model is specified by the following system of Stochastic Differential Equations:

$$
\begin{aligned}
dF(t) &= v(t)C(F(t))dW_1(t) \\
dv(t) &= \gamma v(t)dW_2(t) \\
\langle dW_1(t), dW_2(t)\rangle &= \rho dt \\
F(0) &= f \\
v(0) &= v_0
\end{aligned}
$$

The function *C* is the local volatility. For the standard SABR we have $C(F) := F^{\beta}$

SABR - Approximation Formula
Black Formula with SABR Volatility

An approximation formula for Log-normal (Black) volatilities in the SABR setting:

$$\sigma_{\mathrm{BS}}(K,T) \approx$$

$$\frac{v_0}{(fK)^{\frac{1-\beta}{2}}\left(1+\frac{(1-\beta)^2}{24}\log^2(f/K)+\frac{(1-\beta)^4}{1920}\log^4(f/K)+\dots\right)}\frac{z}{x(z)}$$

$$\left(1+\left(\frac{(1-\beta)^2 v_0^2}{24(fK)^{1-\beta}}+\frac{\rho\beta\gamma v_0}{4(fK)^{\frac{1-\beta}{2}}}+\gamma^2\frac{2-3\rho^2}{24}\right)T+\dots\right),$$

$$z=\frac{\gamma}{v_0}(fK)^{\frac{1-\beta}{2}}\log(f/K), \quad x(z)=\log\left(\frac{\sqrt{1-2z\rho+z^2}+z-\rho}{1-\rho}\right)$$

SABR – Approximation Formula
General Bachelier Formula with SABR Volatility

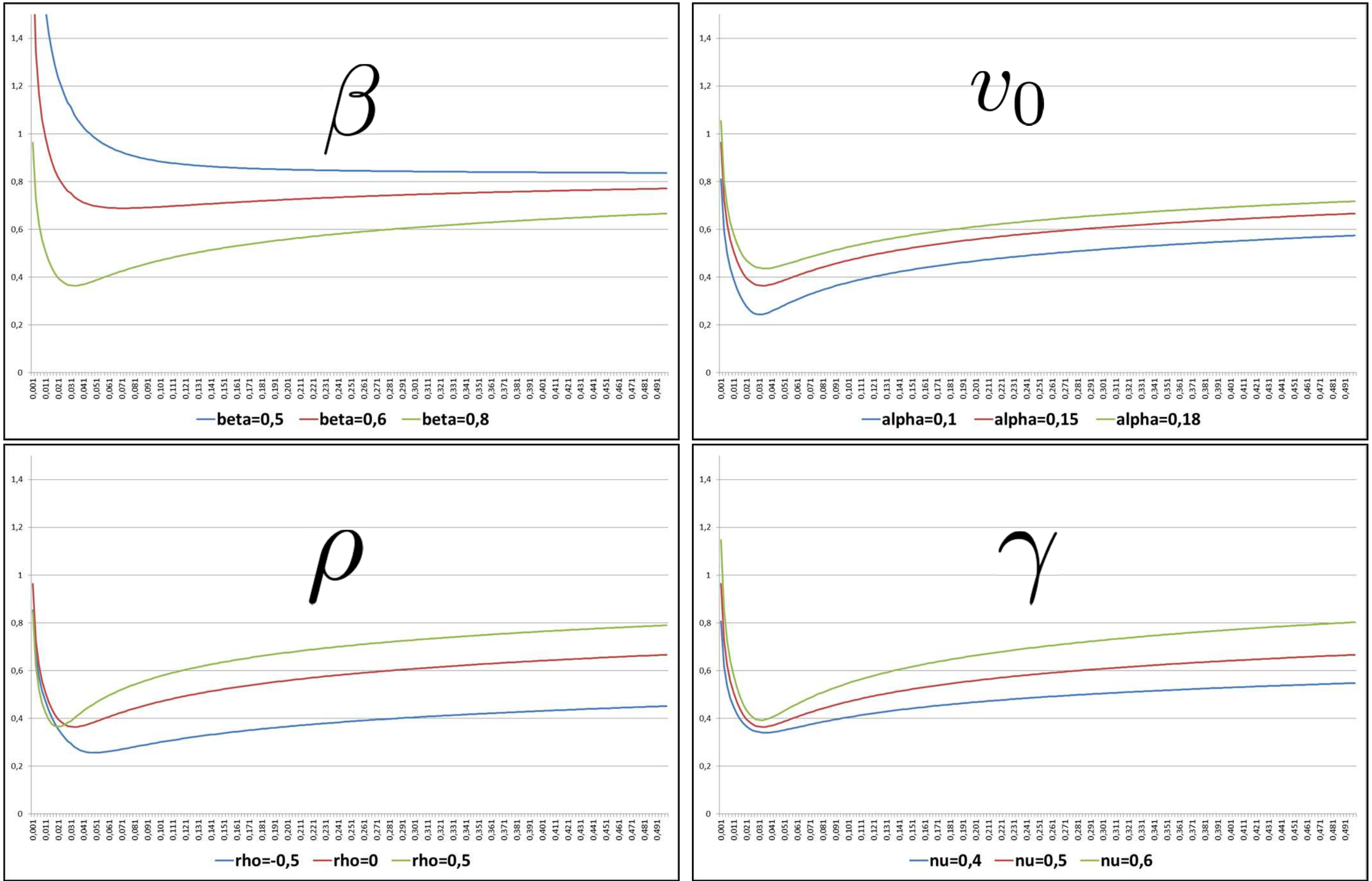
An approximation for the Bachelier (Normal) volatilities in the SABR setting:

$$\sigma_B = \frac{v_0(f-K)}{\int_K^f C(g)^{-1}dg}\left(\frac{\xi}{x(\xi)}\right)\left(1+\left(Gv_0^2+\frac{\rho\gamma v_0}{4}\frac{C(f)-C(K)}{f-K}+\frac{2-3\gamma^2}{24}\right)T+\dots\right)$$

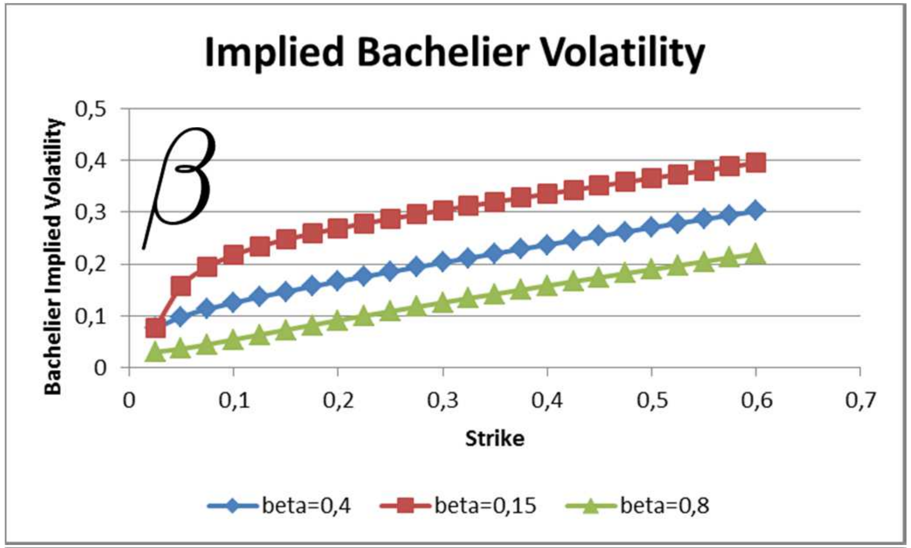
$$\xi := \frac{\gamma}{v_0}\int_K^f C(g)^{-1}dg, \quad x(\xi) := \log\left(\frac{\sqrt{1-2\rho\xi^2}-\rho+\xi}{1-\rho}\right)$$

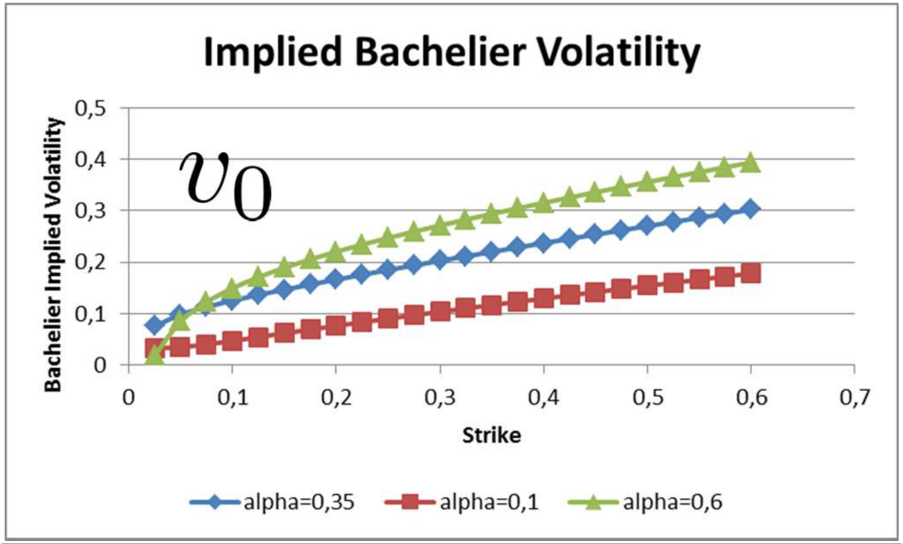
$$G := \log\left(\frac{\int_K^f \frac{C(f)C(K)}{C(g)}dg}{f-K}\right) / \left(\int_K^f C(g)^{-1}dg\right)^2$$

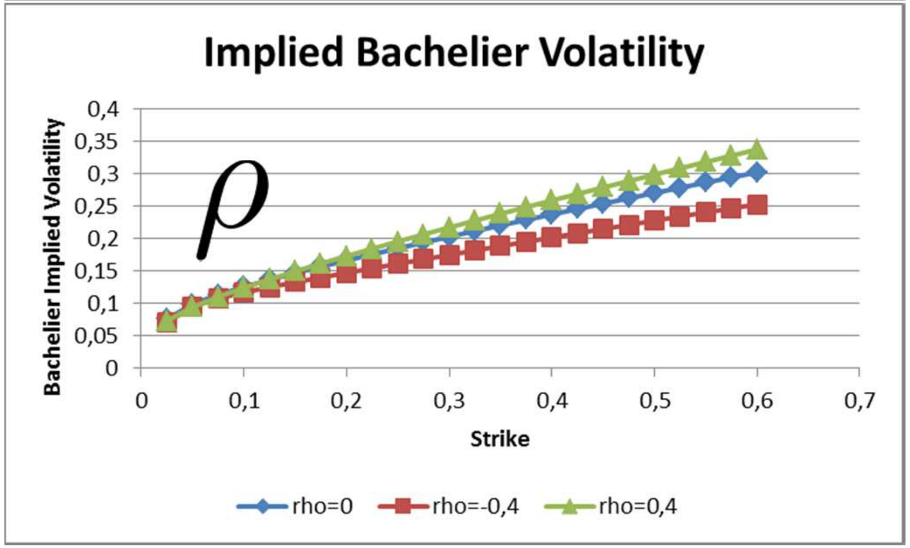
SABR – The SABR Parameters

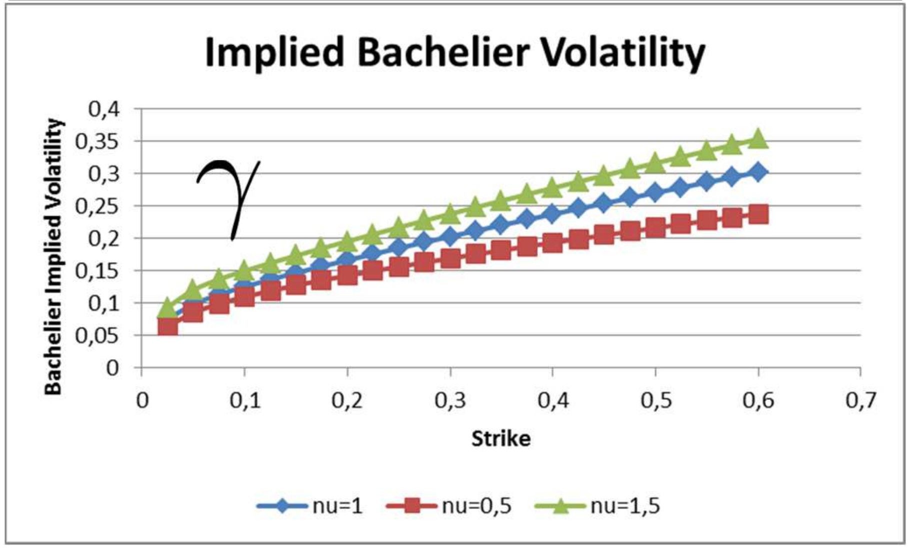


SABR – Die SABR Parameter (Implied Bachelier)



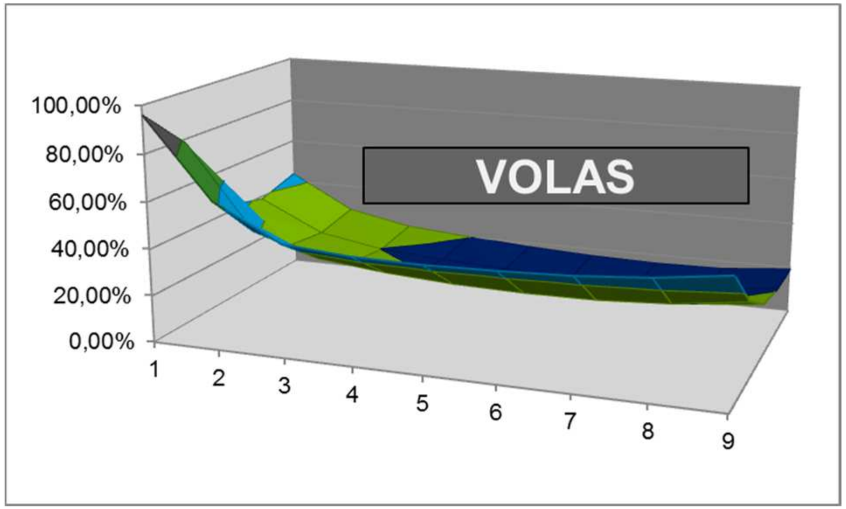


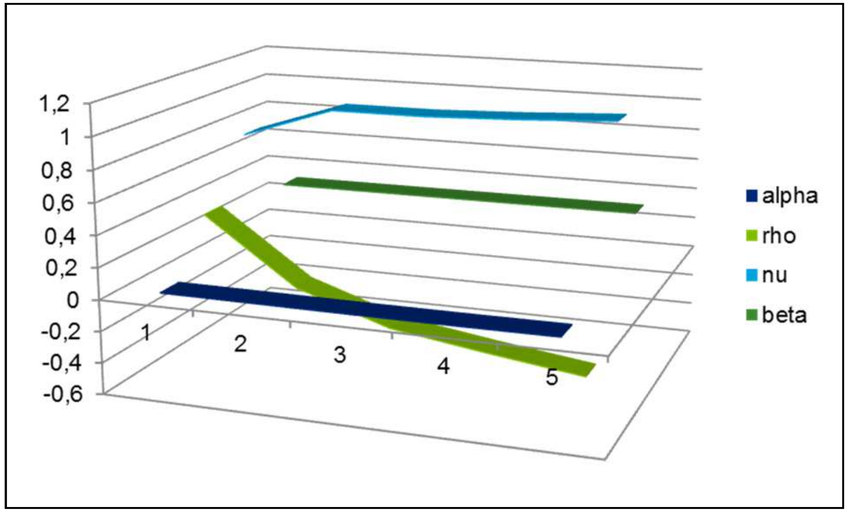


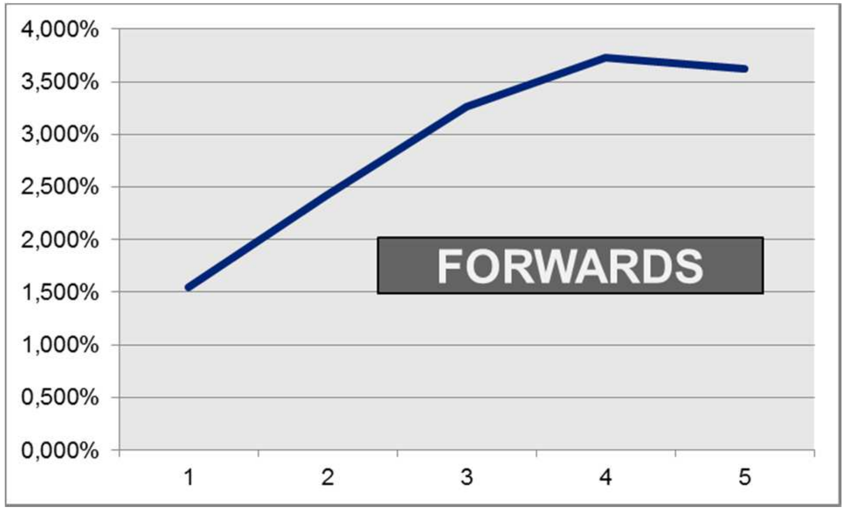


SABR - Calibration

Input

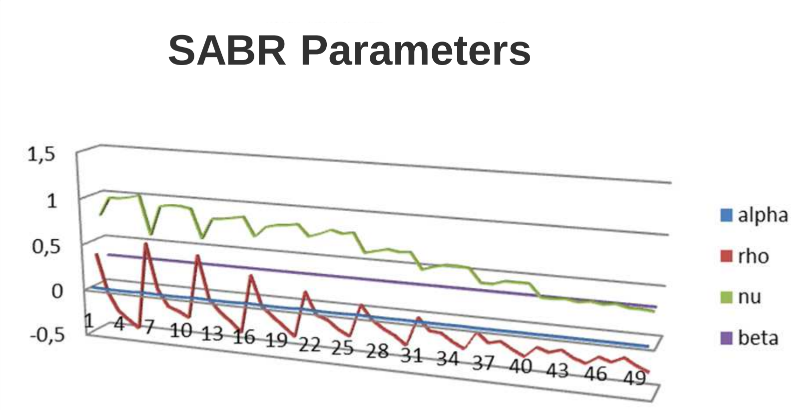




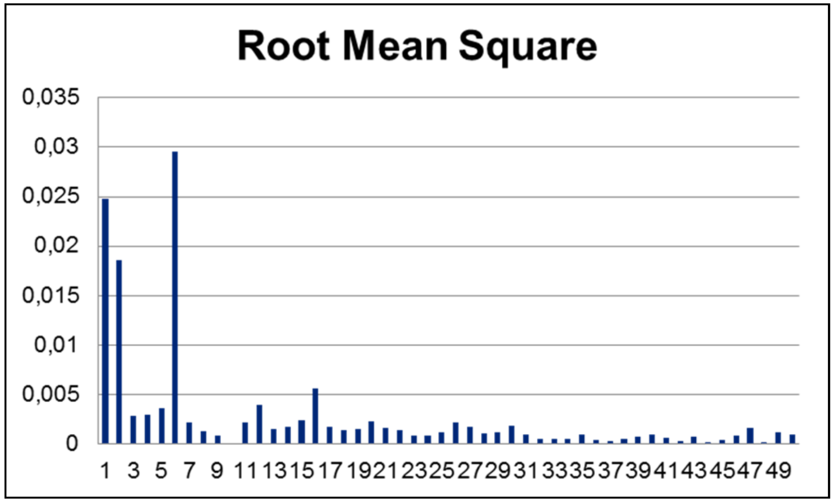


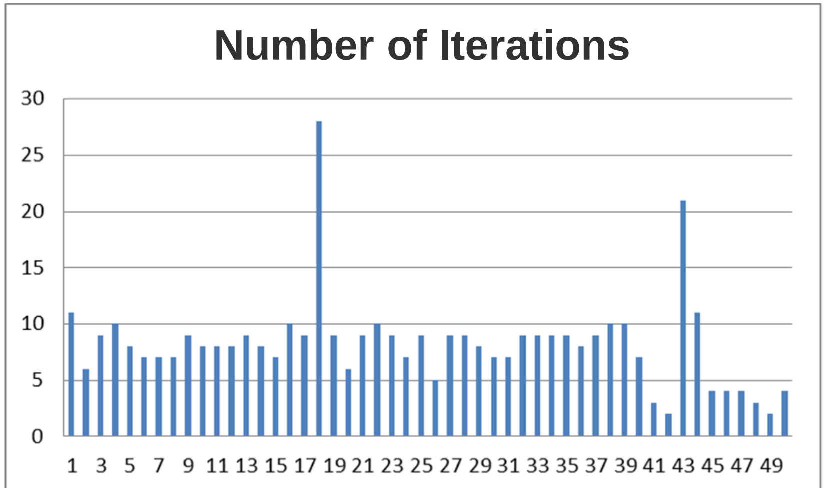
SABR - Calibration

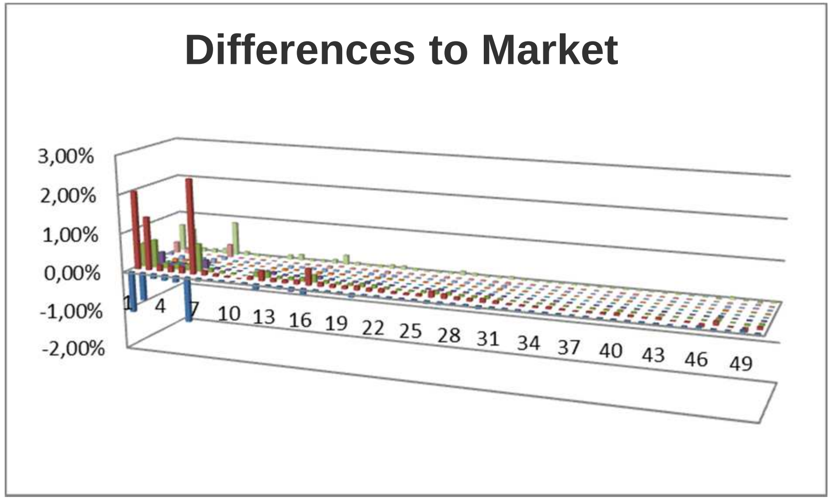
Output



Usually β fix or calibrated using CMS quotes







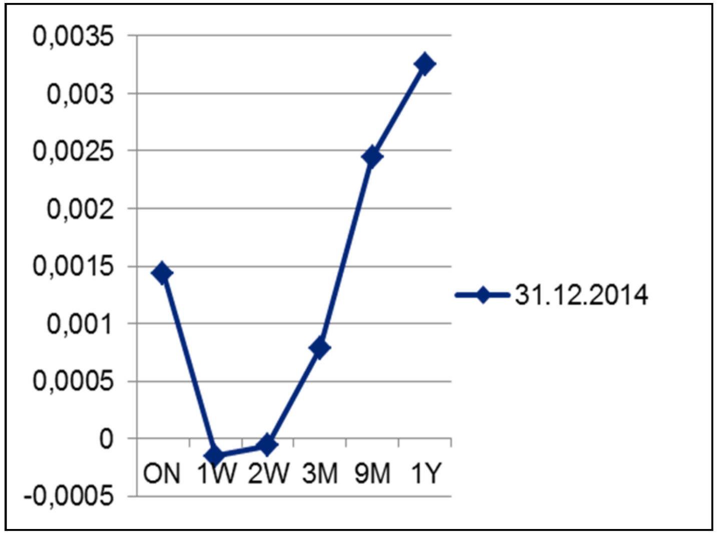
SABR - Calibration

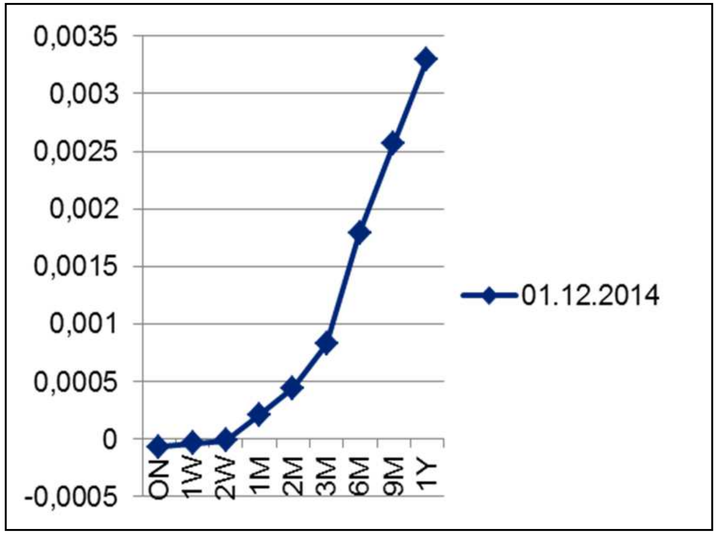
Anything to keep an eye on?

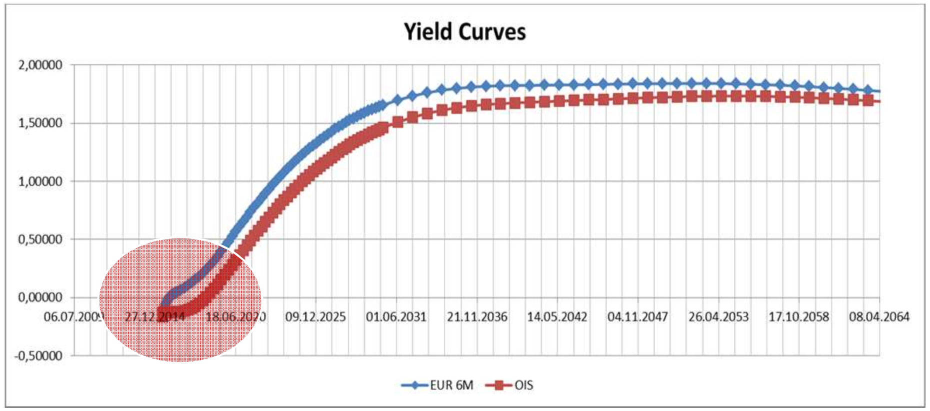
Kontrolle									
0,005	-2,00%	-1,00%	-0,50%	-0,25%	0,00%	0,25%	0,50%	1,00%	2,00%
1m2y	-1,06%	2,02%	0,61%	0,19%	0,00%	-0,01%	0,06%	0,28%	0,69%
1m5y	-0,71%	1,38%	0,72%	0,33%	0,00%	-0,16%	-0,13%	0,11%	0,59%
1m10y	-0,12%	0,15%	0,14%	0,08%	0,00%	-0,04%	-0,05%	0,00%	0,09%
1m20y	-0,14%	0,15%	0,14%	0,09%	0,00%	-0,06%	-0,07%	0,00%	0,10%
1m30y	-0,16%	0,16%	0,16%	0,10%	0,00%	-0,09%	-0,12%	-0,03%	0,13%
3m2y	-1,16%	2,45%	0,71%	0,22%	0,00%	-0,01%	0,08%	0,34%	0,86%
3m5y	-0,07%	0,12%	0,11%	0,05%	0,00%	-0,04%	-0,04%	-0,01%	0,10%
3m10y	-0,05%	0,07%	0,06%	0,03%	0,00%	-0,02%	-0,03%	-0,02%	0,06%
3m20y	-0,04%	0,04%	0,03%	0,02%	0,00%	-0,02%	-0,03%	-0,02%	0,04%
3m30y	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
6m2y	-0,05%	0,09%	0,13%	0,06%	0,00%	-0,03%	-0,04%	-0,01%	0,11%
6m5y	-0,14%	0,27%	0,15%	0,08%	0,00%	-0,05%	-0,07%	-0,02%	0,15%
6m10y	-0,06%	0,09%	0,06%	0,03%	0,00%	-0,02%	-0,04%	-0,03%	0,06%
6m20y	-0,07%	0,09%	0,07%	0,04%	0,00%	-0,04%	-0,05%	-0,03%	0,07%
6m30y	-0,10%	0,11%	0,09%	0,06%	0,00%	-0,05%	-0,08%	-0,06%	0,10%
9m2y	-0,14%	0,43%	0,19%	0,07%	0,00%	-0,04%	-0,04%	0,02%	0,24%
9m5y	-0,06%	0,10%	0,07%	0,04%	0,00%	-0,02%	-0,04%	-0,03%	0,08%
9m10y	-0,05%	0,09%	0,06%	0,03%	0,00%	-0,02%	-0,03%	-0,03%	0,05%
9m20y	-0,06%	0,08%	0,06%	0,03%	0,00%	-0,02%	-0,05%	-0,04%	0,06%
9m30y	-0,10%	0,11%	0,09%	0,05%	0,00%	-0,04%	-0,08%	-0,06%	0,09%
1y2y	-0,03%	0,08%	0,08%	0,04%	0,00%	-0,03%	-0,05%	-0,03%	0,09%
1y5y	-0,05%	0,09%	0,05%	0,02%	0,00%	-0,02%	-0,03%	-0,03%	0,06%
1y10y	-0,03%	0,05%	0,03%	0,01%	0,00%	-0,01%	-0,02%	-0,02%	0,03%
1y20y	-0,04%	0,05%	0,03%	0,02%	0,00%	-0,02%	-0,03%	-0,03%	0,04%
1y30y	-0,05%	0,06%	0,05%	0,02%	0,00%	-0,02%	-0,04%	-0,04%	0,05%

- „Smart Parameters“ (Nice starting values for local solvers) help to stabilize your SABR calibration and reduces the number of iterations which leverages the calibration speed.
- The ATM should be fit perfectly
- Calibration to all EUR Swaption < 0.5 sec. (even in VBA)

SABR
The age of low or even negative rates







Market Quotes

Caps and Floors - ICAP

ICAP — EUR Caps/Floors-Norm Vol OIS — 66) MSG Anbieter — 10:07:47 — Zoom 100%

ICAP Global Menu -> ICAP EMEA -> Interest Rate Options -> IR Options - Digital -> Caps/Floors Vols -> EUR Normal Vols OIS (G...

	Laufzeit	STK	ATM	-0.5	-0.25	-0.13	0.00	0.13	0.25	0.50
1)	1Y		27.2	46.1	35.0	30.0	27.2	31.3	35.0	47.1
2)	18M	0.01	28.9	46.3	36.1	31.5	28.6	32.0	34.9	45.6
3)	2Y	0.03	31.2	46.5	37.5	33.4	30.6	32.9	34.7	43.6
4)	3Y	0.19	39.9	56.7	48.6	44.7	41.1	40.2	39.7	44.0
5)	4Y	0.28	48.1	63.9	56.6	53.1	49.7	48.5	47.7	50.3
6)	5Y	0.39	55.2	68.6	62.3	59.3	56.3	55.0	54.2	55.8
7)	6Y	0.50	60.1	71.2	65.8	63.3	60.8	59.6	58.9	60.0
8)	7Y	0.62	64.1	72.7	68.2	66.0	63.9	62.9	62.3	63.2
9)	8Y	0.72	66.9	73.6	69.8	67.9	66.1	65.2	64.7	65.5
10)	9Y	0.81	68.9	74.0	70.7	69.1	67.5	66.8	66.3	67.1
11)	10Y	0.89	70.4	73.						
12)	12Y	1.02	72.1	73.						
13)	15Y	1.15	72.8	72.						
14)	20Y	1.26	72.3	70.						
15)	25Y	1.28	71.3	68.						
16)	30Y	1.29	70.0	66.						

Negative strikes

(Very) high volatilities

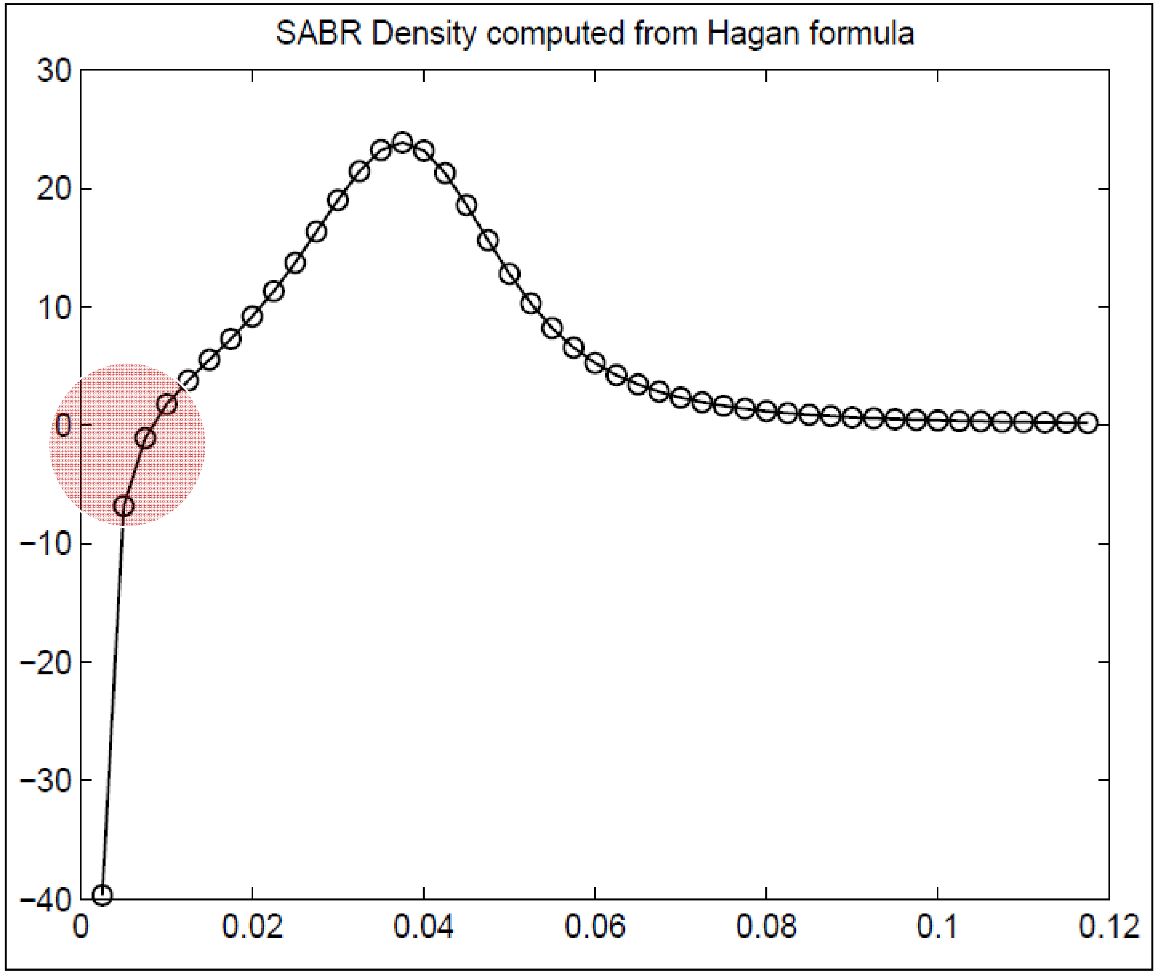
ICAP — EUR Caps/Floors-Norm Vol OIS — 66) MSG Anbieter — 10:13: — Zoom 100%

ICAP Global Menu -> ICAP EMEA -> Interest Rate Options -> IR Options - Digital -> Caps/Floors Vols -> EUR Normal Vols OIS

	Laufzeit	STK	ATM	1.00	1.5	2.00	3.00	5.00	10.00
1)	1Y		27.2	70.5	92.4	113.0	153.0	228.0	402.0
2)	18M	0.01	28.9	66.7	86.7	99.9	142.0	210.0	366.0
3)	2Y	0.03	31.2	62.4	80.4	97.6	130.0	191.0	330.0
4)	3Y	0.19	39.9	57.7	72.9	87.8	116.0	170.0	290.0
5)	4Y	0.28	48.1	59.5	71.6	84.4	109.0	158.0	266.0
6)	5Y	0.39	55.2	62.0	70.7	80.7	102.0	144.0	240.0
7)	6Y	0.51	60.1	64.7	71.0	78.5	95.5	131.0	215.0
8)	7Y	0.62	64.1	67.1	72.1	77.8	91.0	120.0	195.0
9)	8Y	0.73	66.9	68.8	73.0	77.7	88.5	113.0	179.0
10)	9Y	0.82	68.9	70.1	73.7	77.8	87.0	108.0	167.0
11)	10Y	0.90	70.4	71.0	74.4	78.1	86.2	104.0	157.0
12)	12Y	1.03	72.1	71.9	74.8	78.0	84.9	100.0	144.0
13)	15Y	1.16	72.8	71.9	74.6	77.5	84.1	99.4	142.0
14)	20Y	1.26	72.3	71.1	73.4	75.9	81.6	95.0	131.0
15)	25Y	1.28	71.2	70.0	72.1	74.4	79.6	91.9	124.0
16)	30Y	1.29	69.9	68.7	70.7	72.9	77.8	89.4	120.0

Models – An Example for a Possible Pitfall
The SABR Approximation Formula

- After August 2007 several market practices and usances changed significantly.
- Market parameters such as forward swap rates became very small and the corresponding volatilites became very high.
- OIS Discounting and credit/liquidity issues and appearance of multi tenored curves
- We consider the impact on the standard SABR model



Negative Rates
Which models can now be used?

- We have to handle curves
  - Generally the negative rates do not have that impact on curves construction and is a *standard* process these days
  - The appearance of significant basis spreads have much more impact

- We still need to price options
  - Caps/Floors
  - Swaptions
  - CMS Caps/Floors
  - CMS Spread Options

Negative Rates – Models

Pricing Options with negative Strikes

- The displaced diffusion or shifted log-normal model

$$dS(t) = (S+b)\sigma dW(t), \quad S(0) = s_0$$

$$C_{DD}(S(0), K, T, \sigma) \quad = \quad (S(0)+b)\mathcal{N}(d_1) - (K+b)\mathcal{N}(d_2)$$

$$P_{DD}(S(0), K, T, \sigma) \quad = \quad (K+b)\mathcal{N}(-d_2) - (S(0)+b)\mathcal{N}(-d_1)$$

- The Bachelier (or Normal) model

$$dS(t) = \sigma_{\mathrm{N}} dW(t), \quad S(0) = s_0$$

$$C_N(S(0), K, T, \sigma) \quad = \quad (S(0)-K)\mathcal{N}(d(\sigma_{\mathrm{N}})) + \sigma\sqrt{T}n(d(\sigma_{\mathrm{N}}))$$

$$P_N(S(0), K, T, \sigma) \quad = \quad (K-S(0))\mathcal{N}(-d(\sigma_{\mathrm{N}})) + \sigma\sqrt{T}n(d(\sigma_{\mathrm{N}}))$$

As we have already seen there are market quotes

Negative Rates – Models
Pricing Options with negative Strikes

- **The displaced diffusion or shifted log-normal model**

To derive the implied volatility of Black-Scholes or Shifted-LogNormal type the implementation from „Let‘s be rational“ by Peter Jäckel can be used.
A reference implementation is provided (www.jaeckel.org)

- **The Bachelier (or Normal) model**

To derive the implied volatility of Bachelier type the implementation form „Fast and Accurate Analytic Basis Point Volatility“ by Fabien Le Floc'h can be used.
A reference implementation is provided (http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2420757)

Negative Rates - Markets
Implications for a banks business

- Zero Strike Floors (Implicit in many bonds)
- Options with negative strikes
- Model Choice (Construction of Volatility Surfaces/Hedges/Exotics/...)
- IT Systems (Implementation of new models, adjusting existing models)
- Regulations

SABR, No-Arb SABR, Free Boundary SABR

- SDE, Parameters, Numerics -

SABR – Negative Rates
Choosing Parametrizations

Taking again the SABR model:

$$\begin{aligned} dF(t) &= v(t)C(F(t))dW_1(t) \\ dv(t) &= \gamma v(t)dW_2(t) \\ \langle dW_1(t), dW_2(t)\rangle &= \rho dt \\ F(0) &= f \\ v(0) &= v_0 \end{aligned}$$

To be able to handle negative rates we may choose:

$$C(F) = (F+a)^\beta$$ **Displaced SABR**

$$C(F) = 1$$ **Normal SABR**

$$C(F) = |F|^\beta$$ **Free SABR**

SABR – Negative Rates
Methodology

To be able to use the SABR model in a negative rates setting we have to make sure that:

- Efficient and Fast Pricing is possible (e.g. for calibration)
- Monte Carlo Methods for pricing are available (e.g. when combined with a term structure model or a hybrid model)

What are the numerical techniques we have to apply?

- PDE
- Monte Carlo
- Approximation
- Integration

SABR – Integration Formulas (Antonov et al.)

2D Integration 2012 but zero correlation

$$
\begin{aligned}
C(K,T) &= (f-K)^{+} \\
&+ \frac{2}{\pi}\sqrt{Kf}\left(\int_{lb}^{ub}\frac{\sin(|\nu|\Phi(s))}{\sinh(s)}G(\gamma^2 t,s)ds\right. \\
&+ \left.\sin(|\nu|\pi)\int_{lb}^{+\infty}\frac{e^{-|\nu|\Psi(s)}}{\sinh(s)}G(\gamma^2 t,s)ds\right)
\end{aligned}
$$

$$lb = \operatorname{arcsinh}\left(\frac{\gamma|q_K - q_0|}{v_0}\right), \quad ub = \operatorname{arcsinh}\left(\frac{\gamma|q_K + q_0|}{v_0}\right)$$

$$
\begin{aligned}
\Phi(s) &= 2\arctan\left(\sqrt{\frac{\sinh^2(s)-\sinh^2(lb)}{\sinh^2(ub)-\sinh^2(s)}}\right) \\
\Psi(s) &= 2\operatorname{arctanh}\left(\sqrt{\frac{\sinh^2(s)-\sinh^2(ub)}{\sinh^2(s)-\sinh^2(lb)}}\right)
\end{aligned}
$$

$$G(t,s) = 2\sqrt{2}\frac{e^{-t/8}}{t\sqrt{2\pi t}}\int_s^{+\infty}\sqrt{\cosh(u)-\cosh(s)}\,ue^{-\frac{u^2}{2t}}ds$$

$$q_0 = \frac{f^{1-\beta}}{1-\beta}, \quad q_K = \frac{K^{1-\beta}}{1-\beta}, \quad \nu = -\frac{1}{2(1-\beta)}$$

SABR – Integration Formulas (Antonov et al.)
1D Integration 2013 but zero correlation

$$C(K,T) = (f-K)^+ + \mathcal{O}(T,K)$$

$$\mathcal{O}(T,K) = \frac{T^{3/2}}{2\sqrt{2}} \exp\left(-\frac{1}{2}\frac{s_m^2}{\gamma^2 T} - \log\left(\frac{s_m^2}{2\gamma^2}\right) \log\left(K^\beta \sqrt{v_0 v_m} - \mathcal{A}_m\right)\right).$$

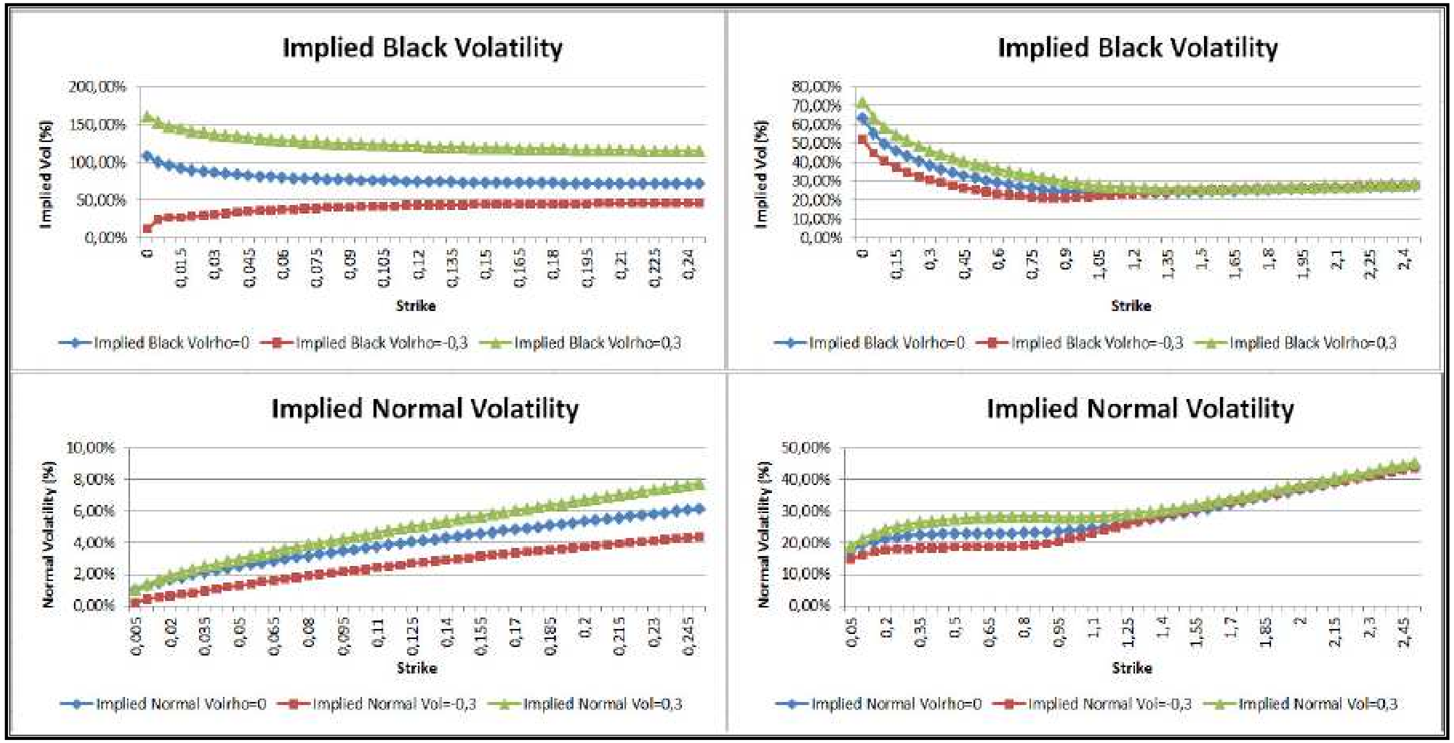
$$G(t,s) \approx \sqrt{\frac{\sinh(s)}{s}} e^{-\frac{s^2}{2t}-\frac{t}{8}} \left(R(t,s) + \delta R(t,s)\right)$$

$$\begin{aligned} R(t,s) &= 1 + \frac{3t\,(s\coth(s)-1)}{8s^2} - \frac{5t^2\left(-8s^2 + 3\,(s\coth(s)-1)^2 + 24\,(s\coth(s)-1)\right)}{128s^4} \\ &+ \frac{35t^3\left(-40s^2 + 3\,(s\coth(s)-1)^3 + 24\,(s\coth(s)-1)^2 + 120\,(s\coth(s)-1)\right)}{1024s^6} \\ \delta R(t,s) &= e^{t/8} - \frac{3072 + 384t + 24t^2 + t^3}{3072} \end{aligned}$$

SABR – Antonov et al. Integration
Example

Set	F	T	γ	β	v_0	ρ
Set 1	0.03	10	0.7	0.5	0.35	$-0.3, 0, 0.3$
Set 2	1	10	1	0.25	0.35	$-0.3, 0, 0.3$

SABR – Antonov et al. Integration
Example



SABR – Integration Formulas (Antonov et al.)

General Case

- Projection onto a Bachelier, CEV or Zero Correlation SABR model

$$\tilde{\nu} = \tilde{\nu}^{(0)} + T\tilde{\nu}^{(1)} + \dots$$

- The projection is not available for all values for the correlation

- No-arbitrage problems (negative values for the density), but much better than for the standard formulas

- The integration limits need to be adjusted wrt to the parameters

- Small values of the CEV parameter lead to wrong results

- Small values for the forward swap rates combined with small values of the CEV parameter can lead to wrong numbers very fast

SABR – Integration Formulas (Antonov et al.)
Projection for a non-zero correlation cases

The authors propose several models for mimicking the original model.

They suggest the zero correlation SABR with parameters $\tilde{\beta}, \tilde{\nu}, \tilde{\gamma}, \tilde{\rho}$

We have for the model parameters:

$$\tilde{\beta} = \beta$$

$$\tilde{\gamma} = \sqrt{\gamma^2 - \frac{3}{2}\left(\rho^2\gamma^2 + \nu\gamma\rho(1-\beta)F^{\beta-1}\right)}$$

$$\tilde{\nu}^{(0)} = \frac{2\theta\Delta\tilde{\gamma}}{\theta^2 - 1} \qquad \Delta = \frac{K^{1-\beta} - F^{1-\beta}}{1-\beta} \qquad \theta = \left(\frac{\nu_{\min} + \rho\nu + \gamma\Delta}{(1+\rho)\nu}\right)^{\frac{\tilde{\gamma}}{\gamma}}$$

$$\left.\frac{\tilde{\nu}^{(1)}}{\tilde{\nu}^{(0)}}\right|_{K=F} = \frac{1}{12}\left(1 - \frac{\tilde{\gamma}^2}{\gamma^2} - \frac{3}{2}\rho^2\right)\gamma^2 + \frac{1}{4}\beta\rho\nu\gamma F^{\beta-1}$$

$$\nu_{\min} = \sqrt{\gamma^2\Delta^2 + 2\rho\gamma\Delta\nu + \nu^2}$$

Normal SABR – Integration Formulas (Korn et al.)

2D Integration C(F)=1 but non-zero correlation

$$C(K,T) = (f_0 - K)^+ + \frac{\sqrt{2}}{\nu\sqrt{1-\rho^2}} \int_b^\infty g(b)h(b)db$$

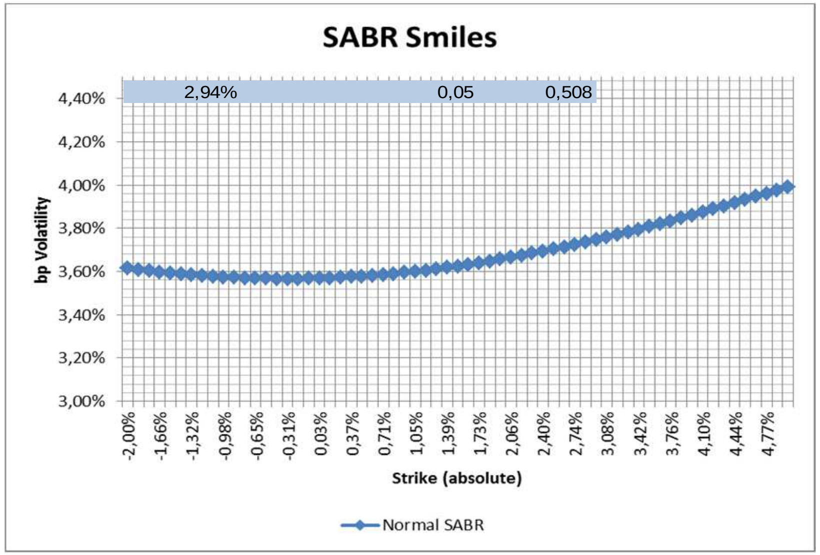
$$h(b) = \int_{a_{min}}^{a_{max}} \frac{1}{\sqrt{\cosh(b) - \cosh(d(a))}} da \qquad d(a) = \cosh^{-1}\left(1 + \frac{N}{2(1-\rho^2 v_0 a}\right)$$

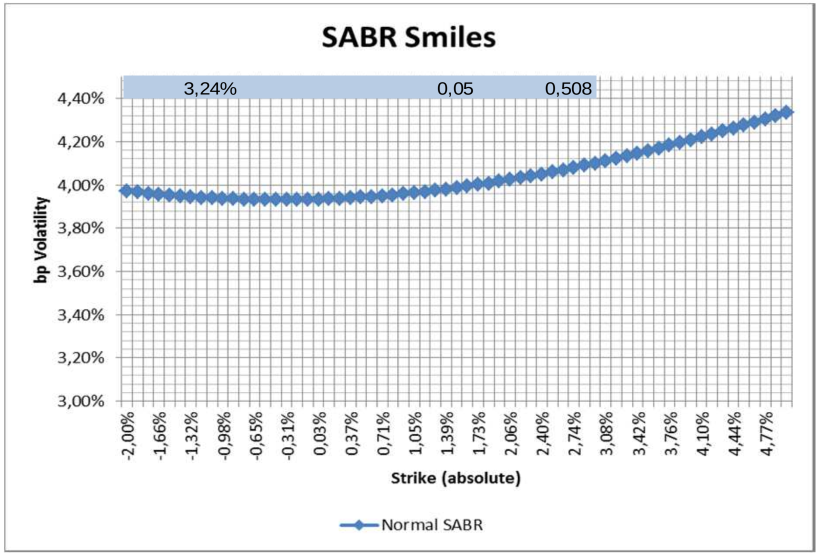
$$a_{min} = \frac{P - \sqrt{Q}}{2}; \quad a_{max} = \frac{P + \sqrt{Q}}{2}$$

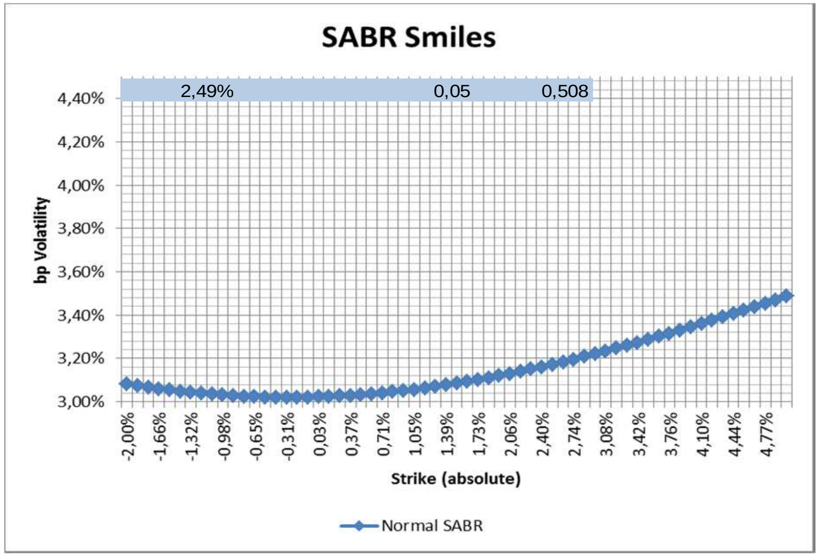
$$\begin{aligned}
N &= (-\gamma m - \rho a + \rho v_0)^2 + (1-\rho^2)(a - v_0)^2 \\
P &= -2m\nu\rho + 2v_0\rho^2 + 2v_0\cosh(b) - 2v_0\rho^2\cosh(b) \\
Q &= 4(-v_0^2 - m^2\gamma^2 + 2mv_0\nu\rho) \\
&\quad + (-2m\gamma\rho + 2v_0\rho^2 + 2v_0\cosh(b) - 2v_0\rho^2\cosh(b))^2 \\
m &= f - K
\end{aligned}$$

Normal SABR

Implied Bachelier Volatilities - alpha

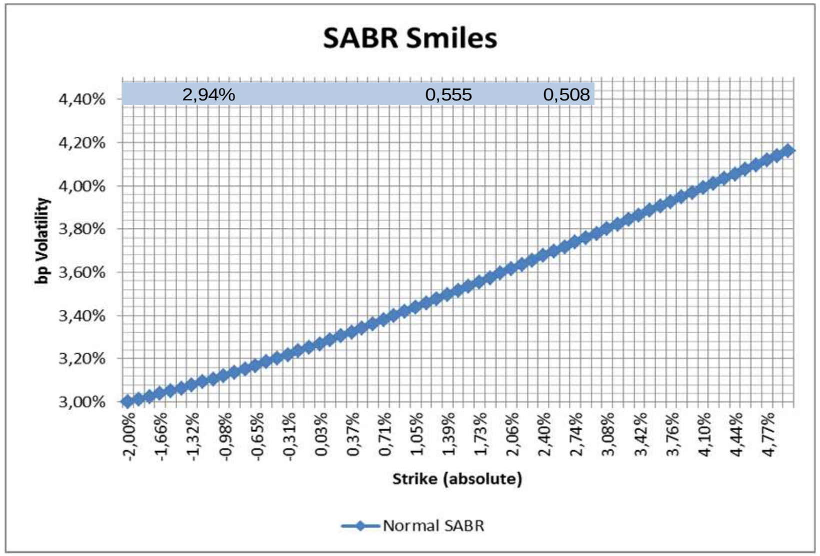


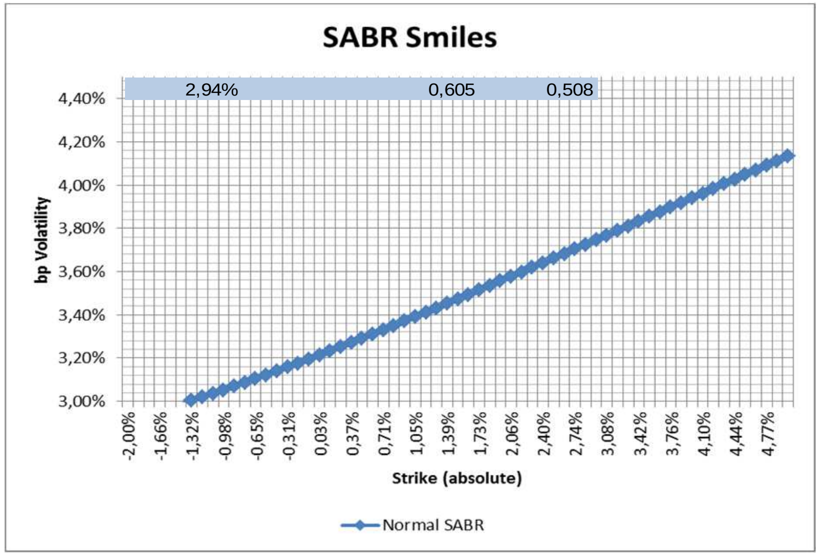


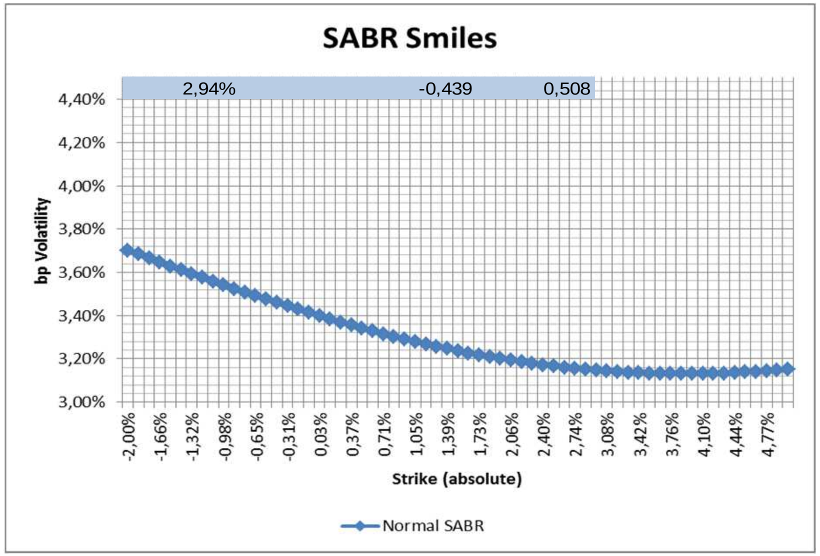


Normal SABR

Implied Bachelier Volatilities - rho

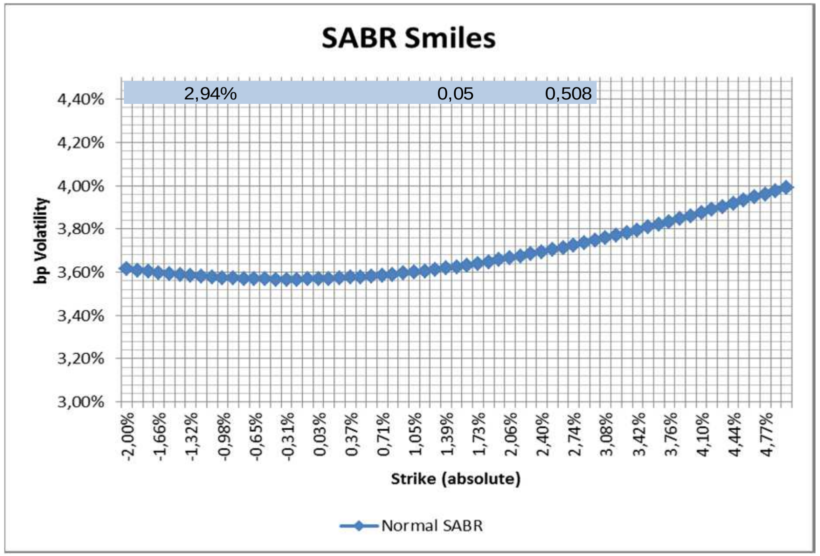


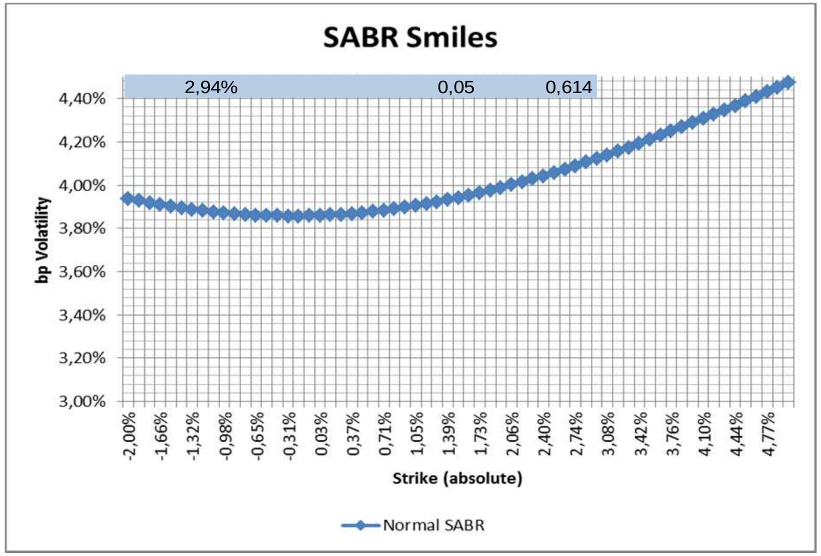


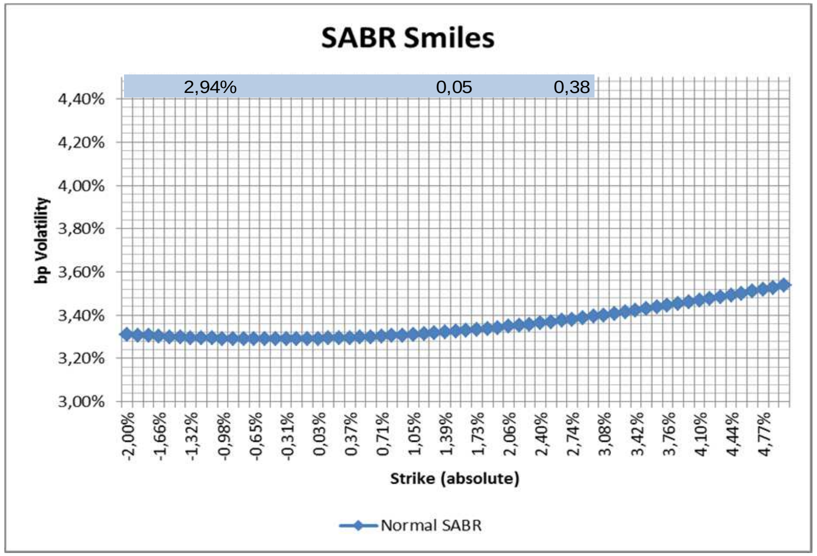


Normal SABR

Implied Bachelier Volatilities - nu







Normal SABR – Integration Formula (Antonov)

1D Integration 2015 and non-zero correlation

$$C(K,T) = (f-K)^{+} + \mathcal{O}(T,K)$$

$$\mathcal{O}(T,K) = \frac{V_0}{\pi}\int_{s_0}^{\infty}\frac{G(\gamma^2 T, s)}{\sinh(s)}\sqrt{\sinh^2(s) - (k-\rho\cosh(s))^2}ds$$

$$\cosh(s_0) = \frac{-\rho k + \sqrt{k^2+\bar{\rho}^2}}{\bar{\rho}^2}$$

$$k = \frac{K-F_0}{V_0} + \rho$$

$$V_0 = \frac{v_0}{\gamma}$$

$$\bar{\rho} = 1-\rho^2$$

No-Arb SABR – PDE Ansatz

SABR becomes technical

- PDE method for calculating the SABR density numerically
- Efficient schemes (DO NOT USE Crank-Nicolson)
- Non-Standard grids and other tricks speed up the calculation
- Density using the PDE and option prices via numerical integration
- (New) approximation formulas for calibration
- Implementation details (http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2402001)

No-Arb SABR – PDE Ansatz
The PDE

- First, we consider the Forward Equation for the reduced density

$$\mathbb{P}\left[F < F_T < F + dF | F_t = f, v_t = v_0\right]$$

- This Forward Equation for the density is then given by

$$\frac{\partial q}{\partial T} = \frac{v^2}{2} \frac{\partial^2 \left[\left(1 + 2\rho\nu y + \nu^2 y^2\right) \exp\left(\rho\nu v \Gamma(F)(T-t)\right) C^2(F) q\right]}{\partial F^2}$$

$$y(F) = \int_f^F \frac{dg}{C(g)} \qquad \Gamma(F) = \frac{C(F) - C(f)}{F - f}$$

- By introducing a function D this simplifies to

$$\frac{\partial q}{\partial T} = \frac{v^2}{2} \frac{\partial^2 \left[D^2(F) q\right]}{\partial F^2}$$

No-Arb SABR – PDE Ansatz
The Boundary Conditions I

- Conservation (Probability Mass is equal to 1)

$$q^L + \int_{F_{\mathrm{Min}}}^{F_{\mathrm{Max}}} q dF + q^R = 1$$

leads to consider

$$0 = \frac{d}{dT}\left(q^L + \int_{F_{\mathrm{Min}}}^{F_{\mathrm{Max}}} q dF + q^R\right)$$

and finally requires

$$\frac{dq^L}{dT} = \lim_{F \downarrow F_{\mathrm{Min}}} \frac{v}{2}\left[D^2(F)q\right]_F \qquad \frac{dq^R}{dT} = -\lim_{F \uparrow F_{\mathrm{Max}}} \frac{v}{2}\left[D^2(F)q\right]_F$$

$$q^L(0) = 0 \qquad q^R(0) = 0$$

No-Arb SABR – PDE Ansatz
The Boundary Conditions II

- Martingale Property (The forward is preserved)

$$\mathbb{E}\left[F_T | F_t = f, v_t = v_0\right] = F_{\text{Min}} q^L(T) + \int_{F_{\text{Min}}}^{\infty} F q D F + F_{\text{Max}} q^R(T) = f$$

$$0 = \frac{d}{dT}\left(q^L + \int_{F_{\text{Min}}}^{F_{\text{Max}}} q dF + q^R\right)$$

Similar to q^L

$$\frac{dq^L}{dT} = \lim_{F \downarrow F_{\text{Min}}} \frac{v}{2}\left[D^2(F)q\right]_F$$ PDE

$$q^L(T) = 0$$

PDE $$\frac{dq}{dT} = \frac{v}{2}\left[D^2(F)q\right]_{FF}$$

leads to the observation that

$$D^2(F)q \to 0, \quad F \downarrow F_{\text{Min}} \qquad D^2(F)q \to 0, \quad F \uparrow F_{\text{Max}}$$

No-Arb SABR – PDE Ansatz

The Final PDE

Continuous part of the distribution

$$\frac{\partial q}{\partial T} = \frac{v^2}{2} \frac{\partial^2 \left[D^2(F) q \right]}{\partial F^2} \qquad q(0) = \delta(F - f)$$

Lower boundary (from model or the modellers decision)

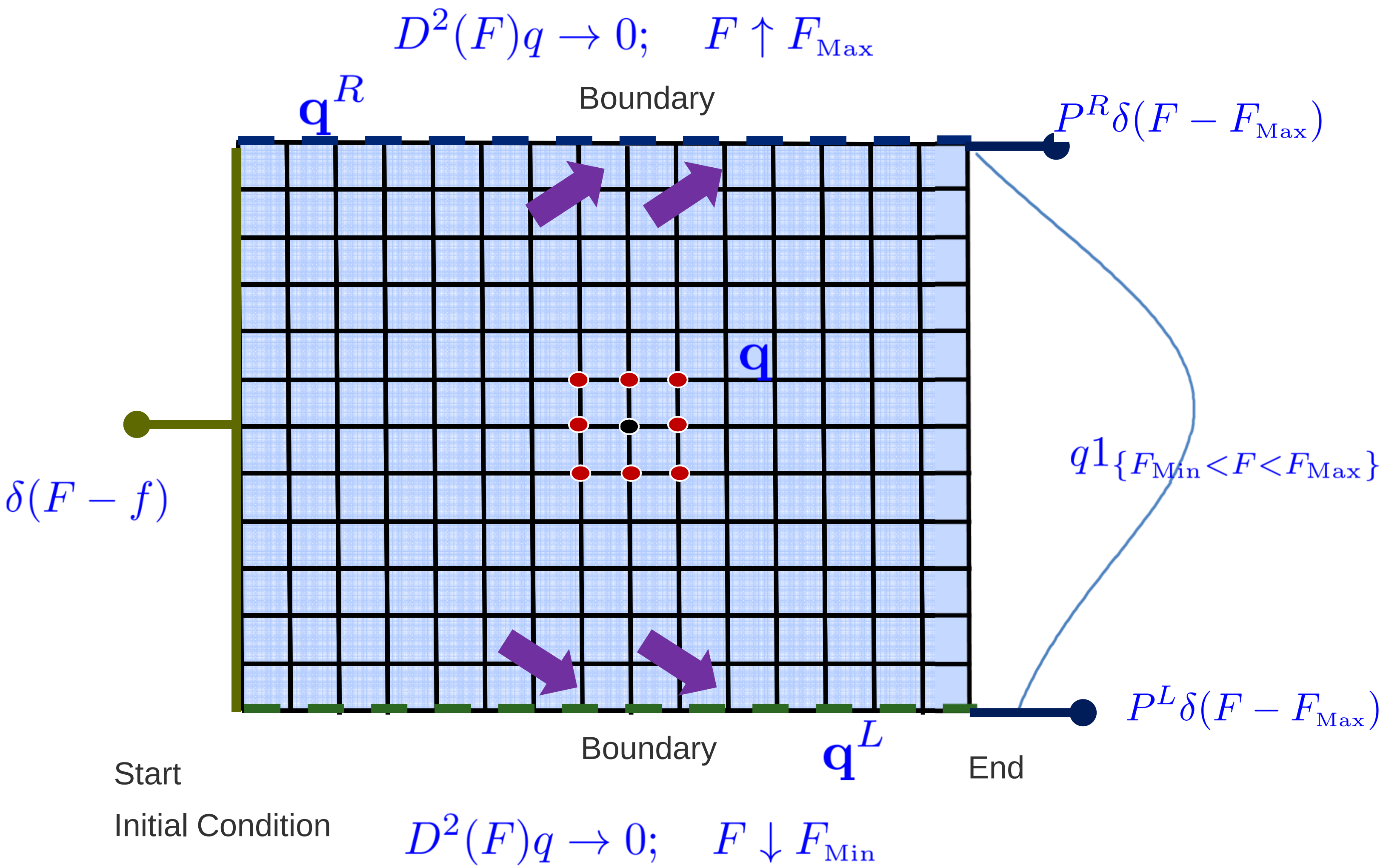
$$\frac{dq^L}{dT} = \lim_{F \downarrow F_{\mathrm{Min}}} \frac{v}{2} \left[D^2(F) q \right]_F \qquad q^L(0) = 0$$

Upper boundary (from discretization)

$$\frac{dq^R}{dT} = - \lim_{F \uparrow F_{\mathrm{Max}}} \frac{v}{2} \left[D^2(F) q \right]_F \quad q^R(0) = 0$$

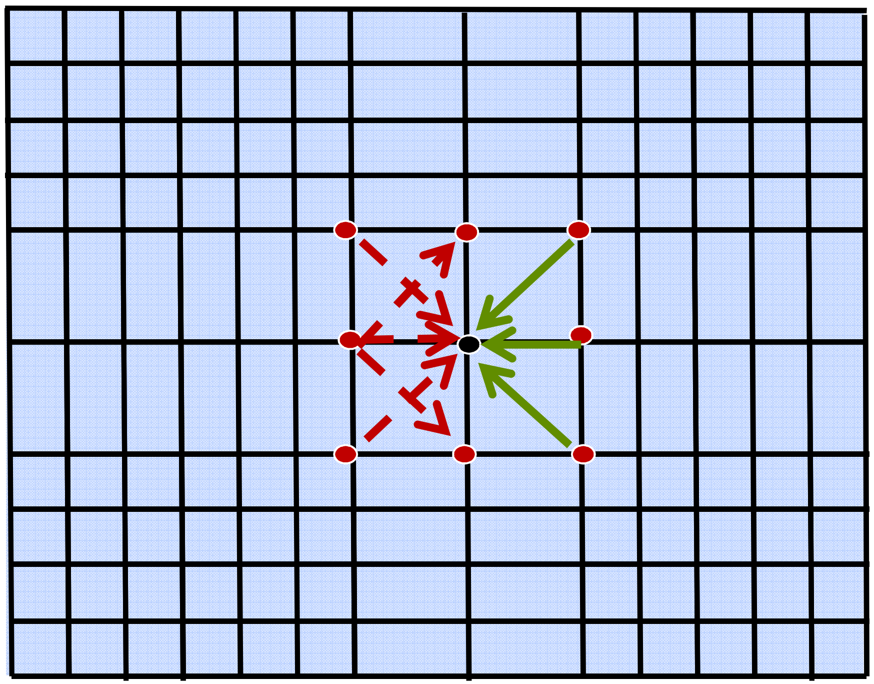
No-Arb SABR – PDE Ansatz

Finite Difference Grid



No-Arb SABR – PDE Ansatz

SABR becomes technical



- Implicit is slow but accurate
- Explicit is fast but may not suitable
- Mixing Implicit and Explicit using the Theta-Scheme (1/2 for Crank-Nicolson)
- How to choose lower and upper values for determning the grid

Standard schemes are tempting to apply but often more efficient/sophisticated methods are available.

Implicit

Solution needs to be calculated by solving a system of equations

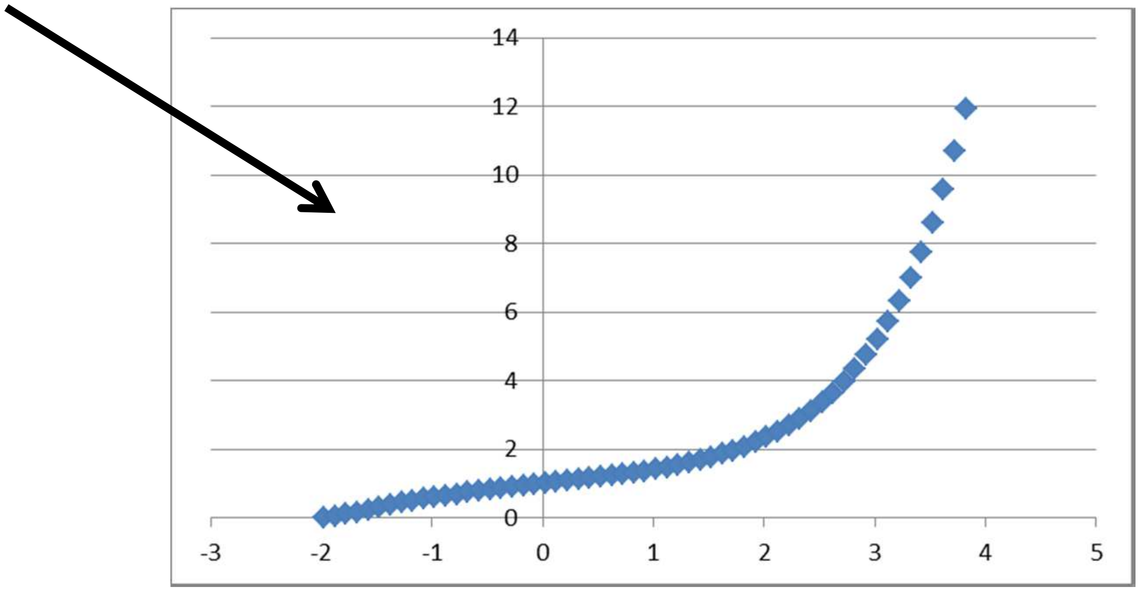
Explicit

Solution is directly given

No-Arb SABR – PDE Ansatz

Details – Transforming Variables (gain efficiency)

f	1			
alpha	0,35			
beta	0,25			
rho	-0,1			
nu	1			
nd	2		ybar	zbar
T	1		-1,33333333	-1,97706584



No-Arb SABR – PDE Ansatz

Discretization

$$
\begin{aligned}
z_j &= z^- + jh \\
y_j &= y(z_j - \frac{h}{2}) \\
F_j &= F(y_j) \\
C_j &= D(F_j) \\
\Gamma_j &= \frac{F_j^\beta - f^\beta}{F_j - f} \\
E_j(T) &= \exp(\rho\nu\alpha\Gamma_j T) \\
t_n &= nT/N \\
q_j^n &= q(z_j, t_n)
\end{aligned}
$$

$$j = 1, \ldots, J, \quad n = 0, \ldots, N-1$$

Between lower and upper bound

Lower bound (z.B. 0 or – Displacement)

$$\frac{\partial \hat{q}}{\partial T}(z, t_n) = L_j^n(z, t_n)$$

$$\frac{\partial P_L}{\partial T}(T) = \frac{C_1}{F_1 - F_0} E_1(T)\hat{q}(z_1, T)$$

$$\frac{\partial P_R}{\partial T}(T) = \frac{C_J}{F_{J+1} - F_J} D_J(T)\hat{q}(z_J, T)$$

Upper bound

No-Arb SABR – PDE Ansatz
The Discrete Operator

$$
\begin{aligned}
L_j^n q(z_j, t_n) \quad = \quad & \frac{1}{\Delta} \frac{C_{j-1}}{F_j - F_{j-1}} E_{j-1}(t_n) q(z_{j-1}, t_n) \\
& - \frac{1}{\Delta} \left(\frac{C_j}{F_{j+1} - F_j} + \frac{C_j}{F_j - F_{j-1}} \right) E_j(t_n) q(z_j, t_n) \\
& + \frac{1}{\Delta} \frac{C_{j+1}}{F_{j+1} - F_j} E_{j+1} q(z_{j+1}, t_n)
\end{aligned}
$$

See LeFloch (2014)

No-Arb SABR – PDE Ansatz

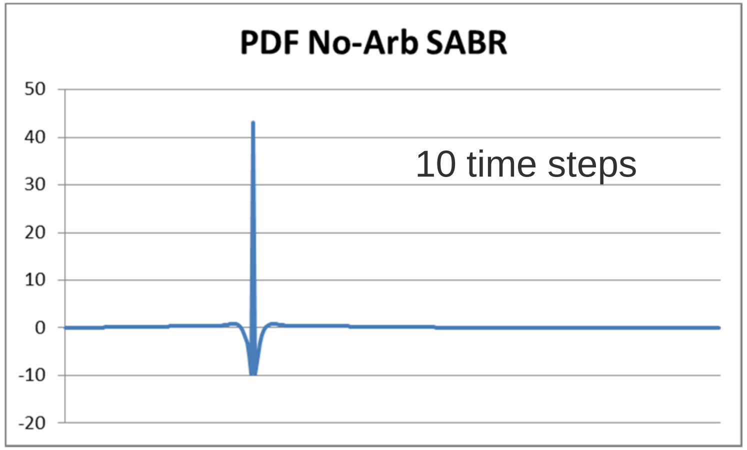
Probabilities at the Boundaries

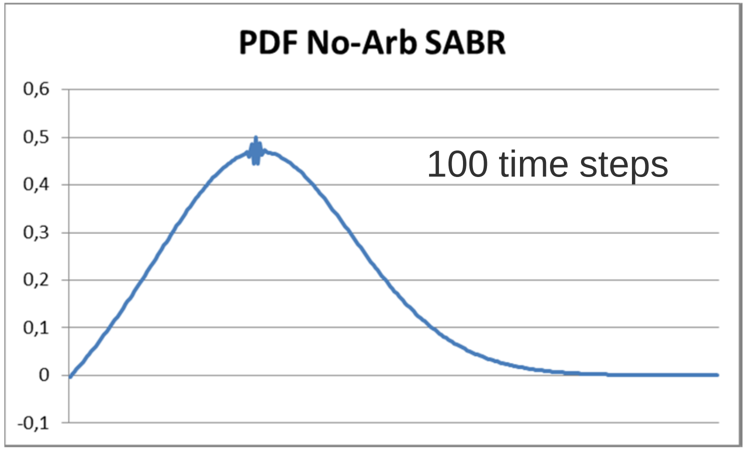
$$\frac{C_0}{F_1 - F_0} E_0(T) q(t_0, T) = -\frac{C_1}{F_1 - F_0} E_1(T) q(z_1, T)$$

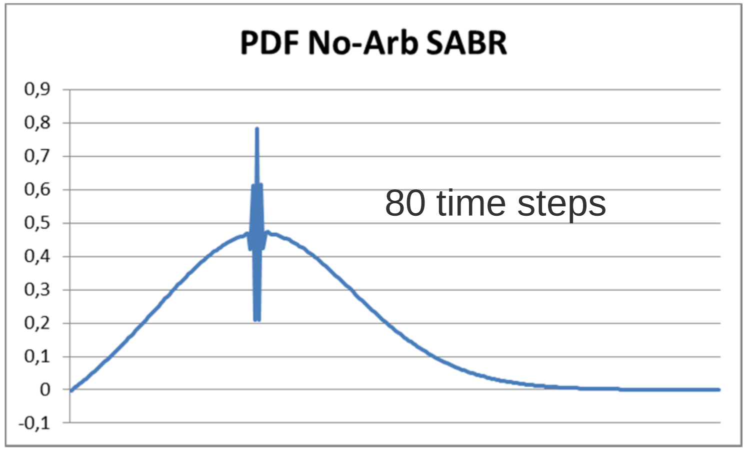
$$\frac{C_{J+1}}{F_{J+1} - F_J} E_{J+1} q(t_{J+1}, T) = -\frac{C_J}{F_{J+1} - F_J} E_J(T) q(z_J, T)$$

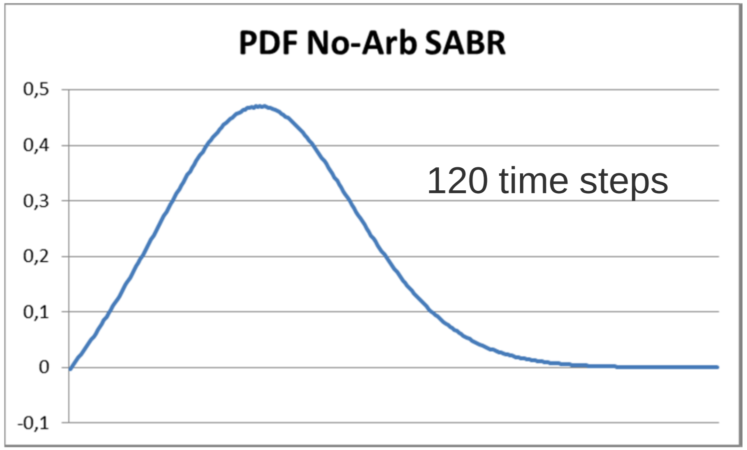
No-Arb SABR – Efficient Schemes

Problems with Crank-Nicolson



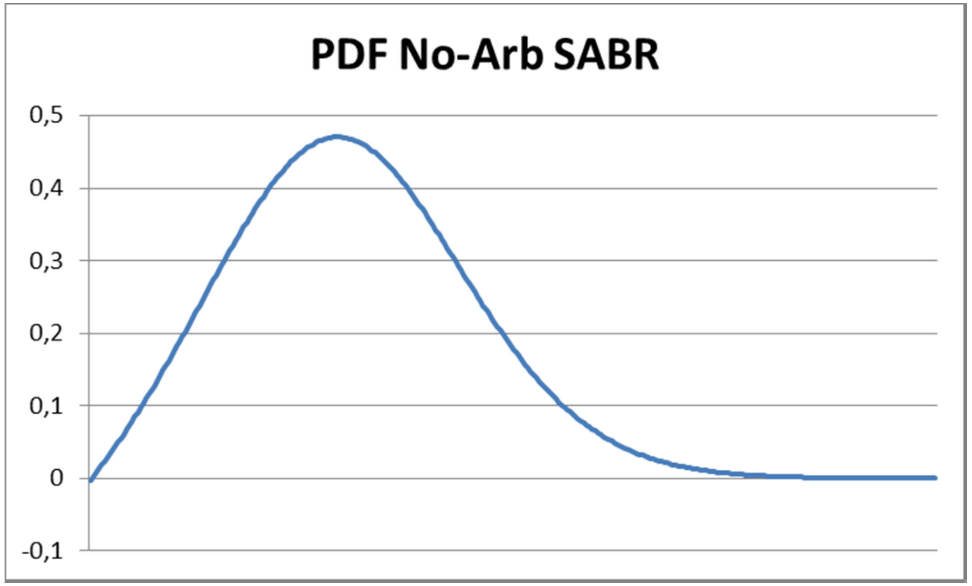






No-Arb SABR – Efficient Schemes

And with an efficient scheme



10 time steps
80 time steps
100 time steps
120 time steps

No-Arb SABR – PDE Ansatz

Standard Schemes

Implicit Euler

$$
\begin{aligned}
q_j^{n+1} - q_j^n &= \Delta L_j^{n+1} q_j^{n+1} \\
P^L(t_{n+1}) - P^L(t_n) &= \Delta \frac{C_1}{F_1 - F_0} E_1(t_{n+1}) q_1^{n+1} \\
P^R(t_{n+1}) - P^R(t_n) &= \Delta \frac{C_J}{F_{J+1} - F_J} E_J(t_{n+1}) q_J^{n+1}
\end{aligned}
$$

Crank Nicolson

$$
\begin{aligned}
q_j^{n+1} - q_j^n &= \frac{\Delta}{2} \left(L_j^{n+1} q_j^{n+1} + L_j^n q_j^n \right) \\
P^L(t_{n+1}) - P^L(t_n) &= \frac{\Delta}{2} \frac{C_1}{F_1 - F_0} \left(E_1(t_{n+1}) q_1^{n+1} + E_1(t_n) q_1^n \right) \\
P^R(t_{n+1}) - P^R(t_n) &= \frac{\Delta}{2} \frac{C_J}{F_{J+1} - F_J} \left(E_J(t_{n+1}) q_J^{n+1} + E_J(t_n) q_J^n \right)
\end{aligned}
$$

No-Arb SABR – PDE Ansatz

Efficient Schemes, e.g. Lawson Swayne

Lawson-Swayne $b = 1 - \frac{\sqrt{2}}{2}$

$$
\begin{aligned}
q_j^{n+b} - q_j^n &= b\Delta L_j^{n+b} q_j^{n+b} \\
P^L(t_{n+b}) - P^L(t_n) &= b\Delta \frac{C_1}{F_1 - F_0} E_1(t_{n+b}) q_1^{n+b} \\
P^R(t_{n+b}) - P^R(t_n) &= b\Delta \frac{C_J}{F_{J+1} - F_J} E_J(t_{n+b}) q_J^{n+b}
\end{aligned}
$$

$$
\begin{aligned}
q_j^{n+2b} - q_j^{n+b} &= b\Delta L_j^{n+2b} q_j^{n+2b} \\
P^L(t_{n+2b}) - P^L(t_{n+b}) &= b\Delta \frac{C_1}{F_1 - F_0} E_1(t_{n+2b}) q_1^{n+2b} \\
P^R(t_{n+2b}) - P^R(t_{n+b}) &= b\Delta \frac{C_J}{F_{J+1} - F_J} E_J(t_{n+2b}) q_J^{n+2b}
\end{aligned}
$$

No-Arb SABR – PDE Ansatz

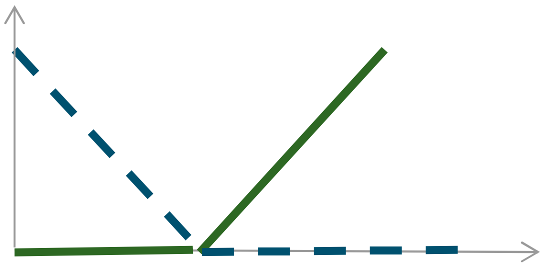
Efficient Schemes

Lawson-Swayne $b = 1 - \frac{\sqrt{2}}{2}$

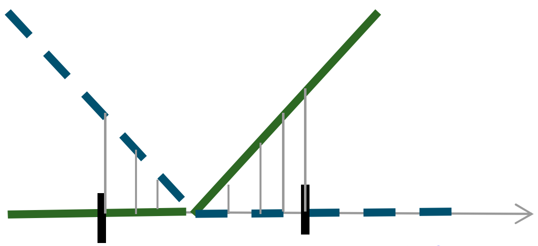
$$
\begin{aligned}
q_j^{n+1} &= (\sqrt{2}+1)q_j^{n+2b} - \sqrt{2}q_j^{n+b} \\
P^L(t_{n+1}) &= (\sqrt{2}+1)P^L(t_{n+2b}) - \sqrt{2}P^L(t_{n+b}) \\
P^R(t_{n+1}) &= (\sqrt{2}+1)P^R(t_{n+2b}) - \sqrt{2}P^R(t_{n+b})
\end{aligned}
$$

No-Arb SABR – Pricing via Integration

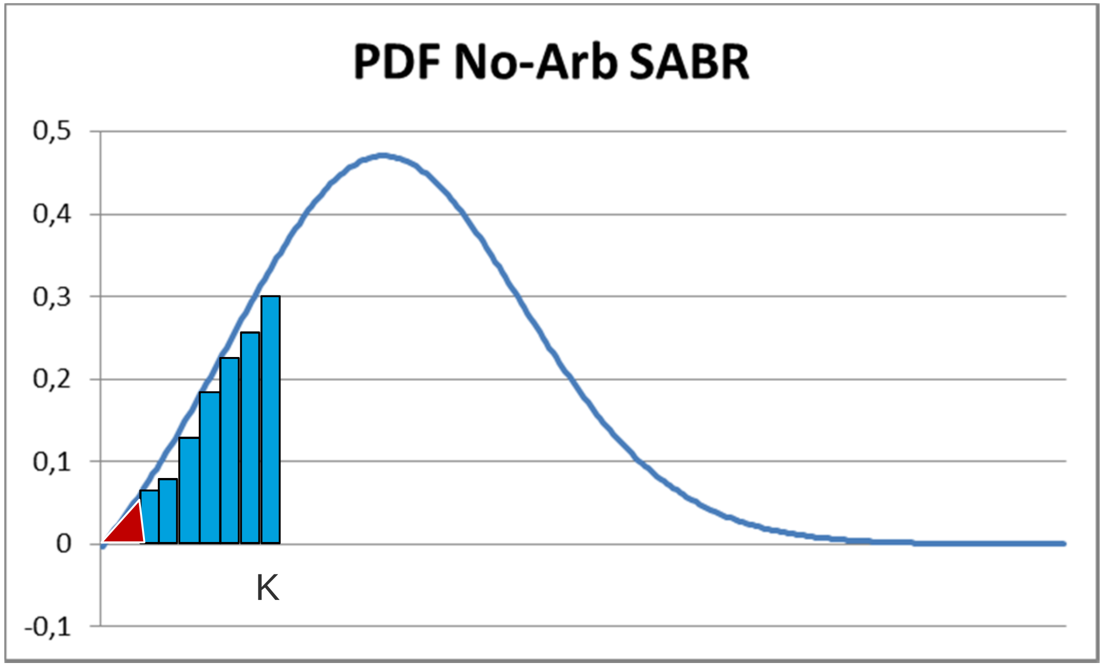
And with an efficient scheme



$$F_{\text{Min}} + (k-1)\Delta < K < F_{\text{Min}} + k\Delta$$



$$F_{\text{Min}} + (k-1)\Delta \qquad F_{\text{Min}} + k\Delta$$



Integrating $\begin{matrix}(F-K)^+ \\ (K-F)^+\end{matrix}$

with respect to the density (blue)

- Choose an integration scheme
- Just use the representation of the PDE with appropriate handling of the left tail (red)

No-Arb SABR – Pricing via Integration

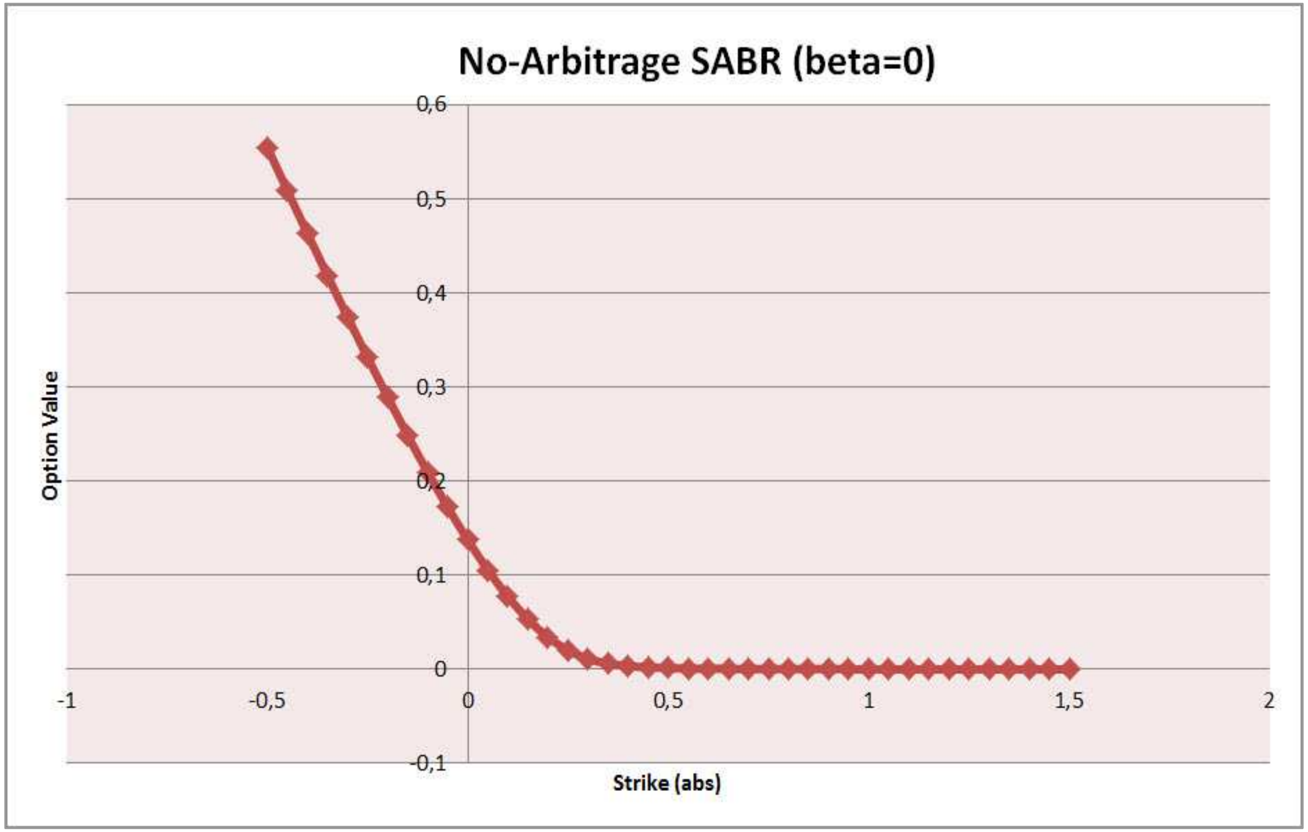
Option Pricing Formulas

We assume that the FDM Scheme applied uses N steps and a size of Δ

$$F_{\text{Min}} + (k-1)\Delta < K < F_{\text{Min}} + k\Delta$$

$$\begin{aligned} V_{\text{Call}}(T,K) &= \frac{1}{2}\left[F_{\text{Min}} + k\Delta - K\right]^2 q_k^N \\ &+ \sum_{j=k+1}^{N} \left[F_{\text{Min}} + \left(j - \frac{1}{2}\right)\Delta - K\right] \Delta q_j^N \\ &+ \left[F_{\text{Max}} - K\right] q^R \\ V_{\text{Put}}(T,K) &= \frac{1}{2}\left[K - F_{\text{Min}} - (k-1)\Delta - K\right]^2 q_k^N \\ &+ \sum_{j=1}^{k} \left[K - F_{\text{Min}} - \left(j - \frac{1}{2}\right)\Delta\right] \Delta q_j^N \\ &+ \left[K - F_{\text{Min}}\right] q^L \end{aligned}$$

No-Arb SABR – PDE Ansatz
SABR becomes technical



No-Arb SABR – Approximation Formulas
Old Results from Libor Market Models

- Andersen/Brotherton-Ratcliffe show an approximation in the context of Libor Market Models with Stochastic Volatility
- The results can be carried over to the SABR model

$$\sigma(T,K) = \frac{\Sigma_0(K)u^{1/2}(T) + \Sigma_1(K)u^{3/2}(T)}{\sqrt{T}}$$

$$u(T) := T + \frac{1}{2}\rho\nu\alpha\Gamma(K)T^2 + \mathcal{O}(T^3)$$

$$\Gamma(K) = \frac{(K+b)^\beta - (F+b)^\beta}{K-F}$$

To be determined for SABR!

- This approximation can be used in calbration

No-Arb SABR – Approximation Formulas
Old Results from Libor Market Models

LogNormal Volatility Approximation

$$
\begin{aligned}
\sigma_{LN}^{AB} &= \frac{1}{(\xi(K))} \log\left(\frac{f+b}{K+b}\right)\left[1+\left(g_{LN}(K)) + \frac{1}{4}\rho\nu\alpha\Gamma(K)\right)T\right] \\
g_{LN}(K) &= -\frac{1}{x(\xi(K))^2} \log\left(\frac{\log\left(\frac{\frac{f+b}{K+b}}{x(\xi(K))}\right)}{x(\xi(K))}\sqrt{\frac{(f+b)(K+b)}{D(f)D(K)}}\right) \\
D(K) &= \sqrt{\alpha^2 + 2\alpha\rho\nu y(K) + \nu^2 y(K)^2} K^\beta \\
\Gamma(K) &= \frac{(K+b)^\beta - (f+b)^\beta}{K-f} \\
y(K) &= \frac{(K+b)^{1-\beta} - (f+b)^{1-\beta}}{1-\beta}
\end{aligned}
$$

Bachelier Volatility Approximation

$$
\begin{aligned}
\sigma_{N}^{AB} &= \frac{f-K}{(\xi(K))}\left[1+\left(g_N(K)) + \frac{1}{4}\rho\nu\alpha\Gamma(K)\right)T\right] \\
g_N(K) &= -\frac{1}{x(\xi(K))^2} \log\left(\frac{f-K}{x(\xi(K))\sqrt{D(f)D(K)}}\right) \\
x(\xi(K)) &= \frac{1}{\nu} \log\left(\frac{\sqrt{1-2\rho\xi(K)+\xi(K)^2} - \rho + \xi(K)}{1-\rho}\right) \\
\xi(K) &= \frac{\nu}{\alpha(1-\beta)}\left((f+b)^{1-\beta} - (K+b)^{1-\beta}\right)
\end{aligned}
$$

Comparison – Approximation vs Integration/PDE
Benchmarking Approximation Formulas

- Comparison of approximation formulas
  - Comparison to Bachelier volatility
  - Comparison to Call prices

- Benchmark against Integration/PDE Solutions
  - Integration/PDE leads to density and prices
  - Implied Bachelier Vol Solver is used (-> see later)

Comparison –Approximation Formulas for SABR

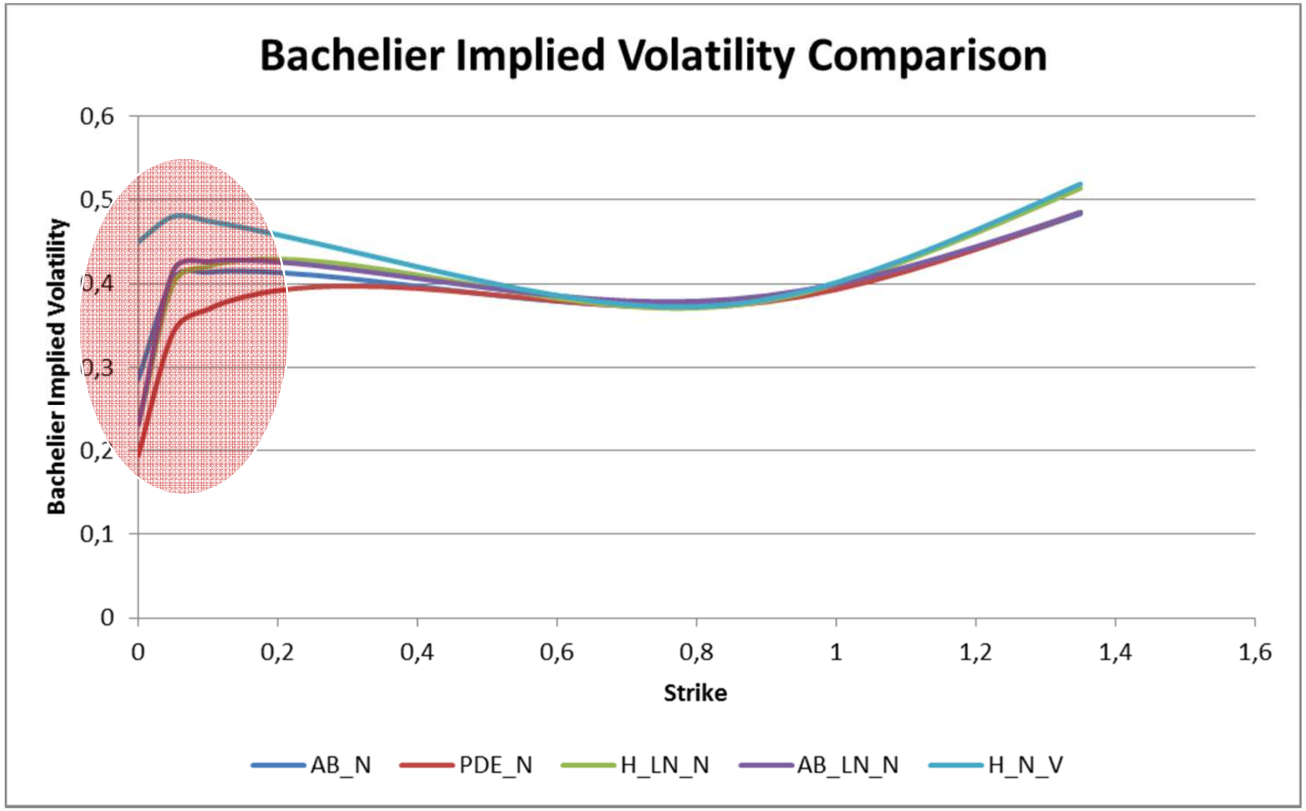
Hagan, Andersen/Brotherton-Ratcliffe Approxformulas vs PDE Solution

forward	1
beta	0,25
v0/alpha	0,35
gamma/nu	1
rho	0,35
tau	2
displacement	0

Values from

Hagan, P.

ICBI Global Derivatives 2014

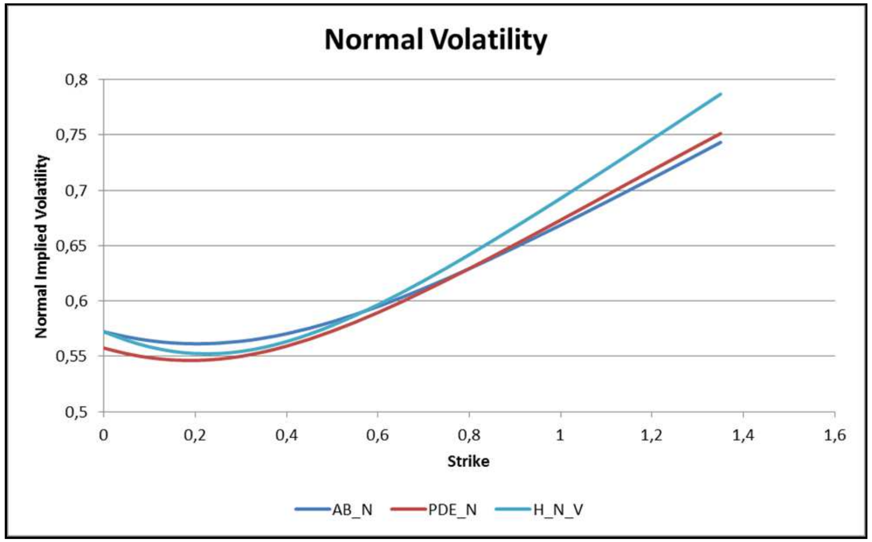


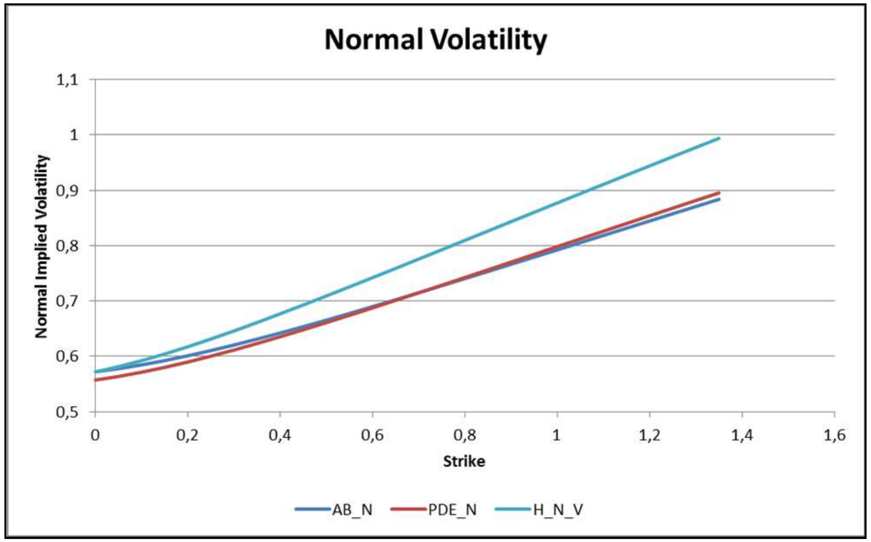
Comparison – Normal Vol Approximation vs PDE

Hagan, Andersen/Brotherton-Ratcliffe Approxformulas vs PDE Solution

forward	**0,0005**
beta	**0**
v0/alpha	**0,5**
gamma/nu	**1**
rho	**-0,3**
tau	**2**
displacement	**0**

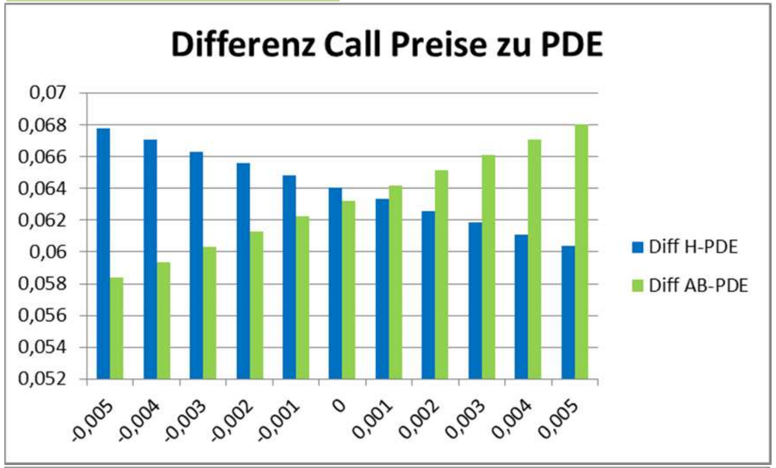
forward	**0,0005**
beta	**0**
v0/alpha	**0,5**
gamma/nu	**1**
rho	**0,3**
tau	**2**
displacement	**0**

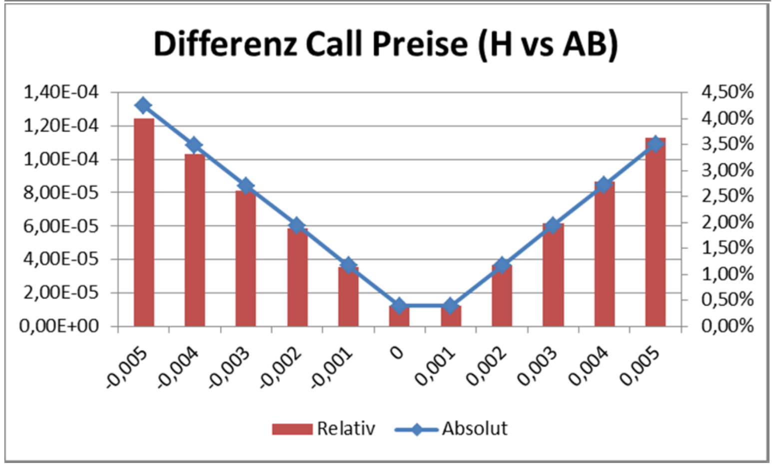




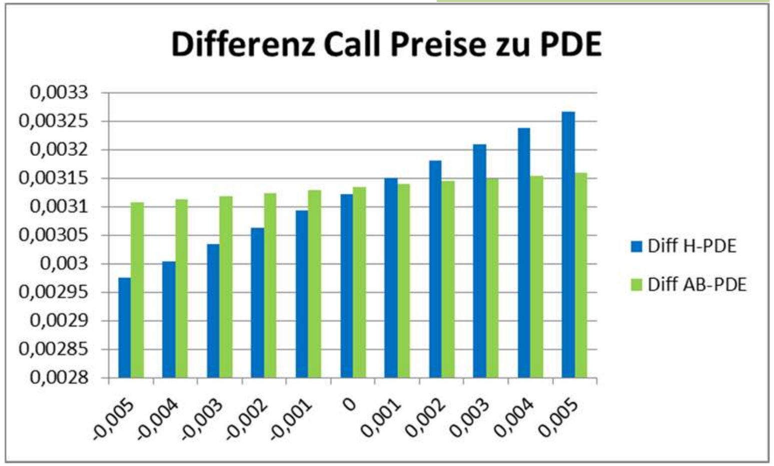
Comparison – Approximations H vs AB

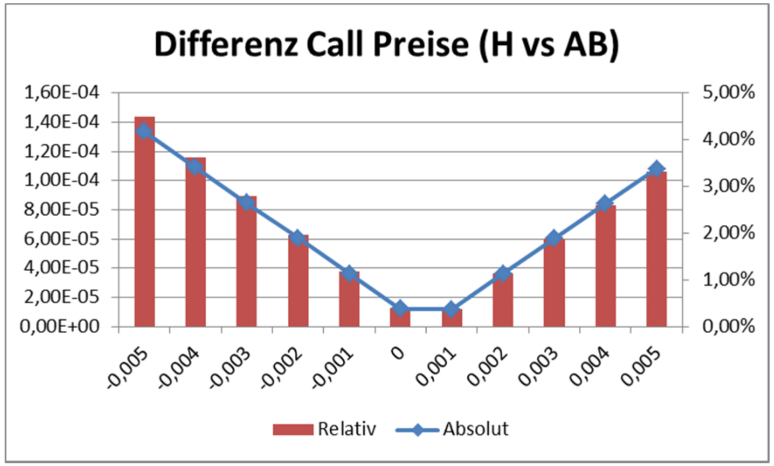
forward (f0)	**0,0005**
beta	**0**
v0/alpha	**0,5**
gamma/nu	**1**
rho	**-0,9**
T	**2**





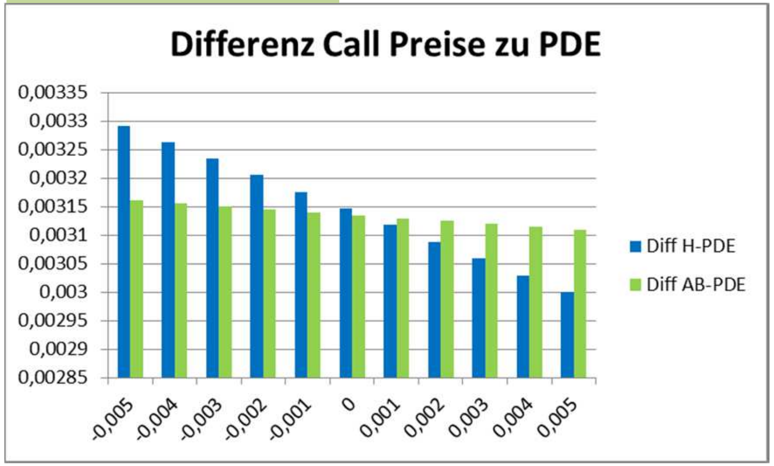
forward (f0)	**0,0005**
beta	**0**
v0/alpha	**0,5**
gamma/nu	**1**
rho	**0,9**
T	**2**

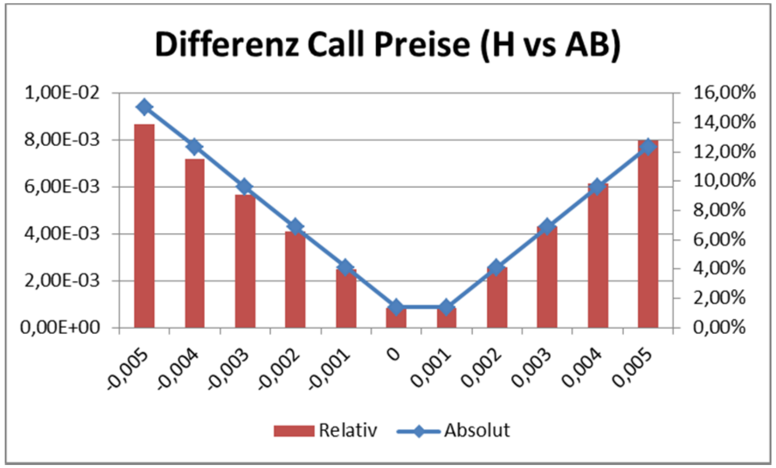




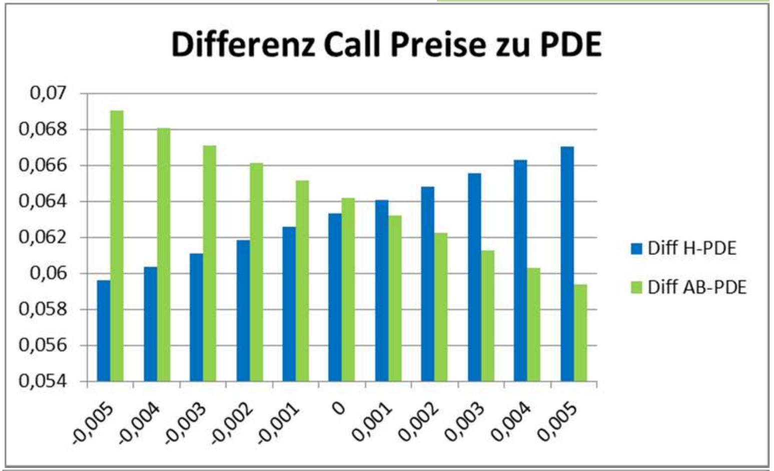
Comparison – Approximations H vs AB

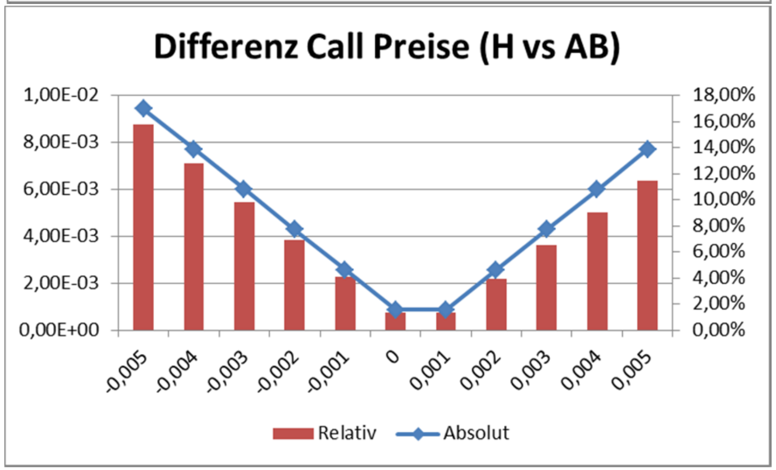
forward (f0)	**0,0005**
beta	**0**
v0/alpha	**0,5**
gamma/nu	**1**
rho	**-0,6**
T	**20**





forward (f0)	**0,0005**
beta	**0**
v0/alpha	**0,5**
gamma/nu	**1**
rho	**0,6**
T	**20**





Comparison – Approximations H vs AB
Conclusion

- Numerical experiments suggest that the correlation determines which approximation to apply (Hagan approximation or Andersen/Brotherton-Ratcliffe approximation)
- The calculation time does not differ between the methods
- We need to validate the calibration procedure
  (Do the parameters run to the boundaries?)
- The approximation formulas lead to better results than the PDE pricer
  - PDE approach is numerically much more involved
  - We need methods to stabilize the set-up

Free Boundary SABR – (Antonov et al.)

SABR gets technical

There is an approximation formula which numerically approximates the free boundary SABR model with zero correlation:

$$
\begin{aligned}
dS &= \alpha |S|^{\beta} dW_1(t) \\
d\alpha &= \nu \alpha dW_2(t) \\
\langle dW_1(t), dW_2(t) \rangle &= \rho dt
\end{aligned}
$$

For the general model a Markovian projection has to be applied to approximate this case by means of the zero correlation model.

We have a PDE solution and approximation formulas here as well, see

Approximating Solution for the Free Boundary SABR for Pricing and Calibration

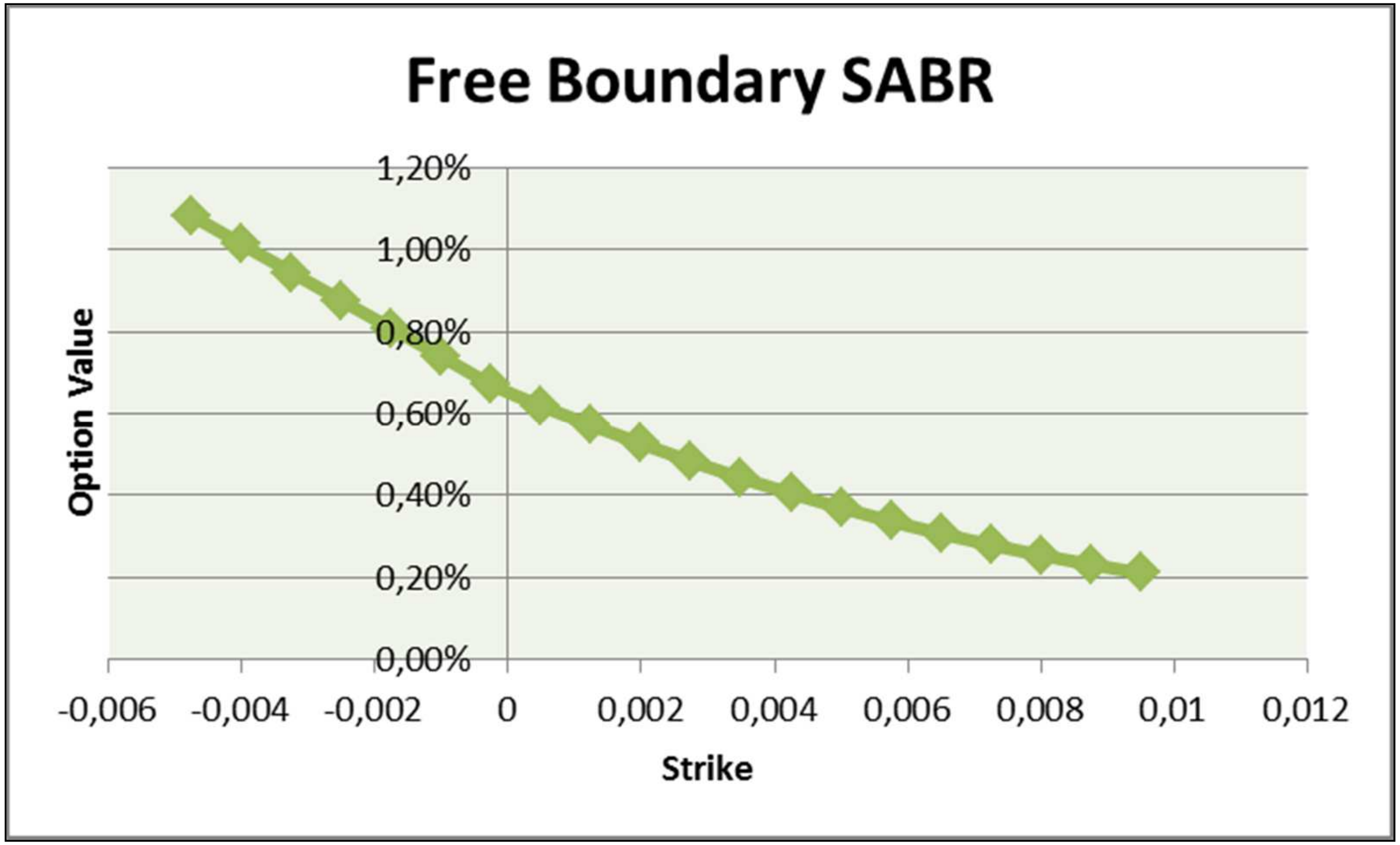
Joerg Kienitz
Deloitte & Touche GmbH

August 19, 2015

http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2647344

Free Boundary SABR - Example

Calculation via Numerical Integration



Free Boundary SABR – Approximation I

Approximation in Free SABR Setting

$$\sigma_B^H(T,K) \approx \frac{\alpha(f-K)}{\int_K^f \frac{ds}{C(s)}} \left(\frac{\xi}{x(\xi)}\right) \left(1 + \left(g\alpha + \frac{\rho\nu\alpha}{4}\frac{C(f)C(K)}{f-K} + \frac{2-3\rho^2}{24}\nu^2\right)T\right)$$

For the free SABR case we find:

$$I := \int_K^f |x|^{-\beta} dx \qquad I = \begin{cases} \frac{(-f)^{1-\beta} - (-K)^{1-\beta}}{1-\beta} & K < 0, f < 0 \\ \frac{f^{1-\beta} + (-K)^{1-\beta}}{1-\beta} & K < 0, f > 0 \\ \frac{f^{1-\beta} - K^{1-\beta}}{1-\beta} & K > 0, f > 0 \end{cases}$$

Free Boundary SABR – Approximation II

Approximation in Free SABR Setting

$$\sigma_N^{AB}(T,K) = \begin{cases} \frac{|f|+|K|}{\xi_K}\left(1+(g_K+\frac{1}{4}\rho\nu\alpha\Gamma_K)T\right) & K<0, F\geq 0 \\ \frac{|f|-|K|}{\xi_{KK}}\left(1+(g_K+\frac{1}{4}\rho\nu\alpha\Gamma_K)T\right) & \text{else} \end{cases}$$

$K < 0$

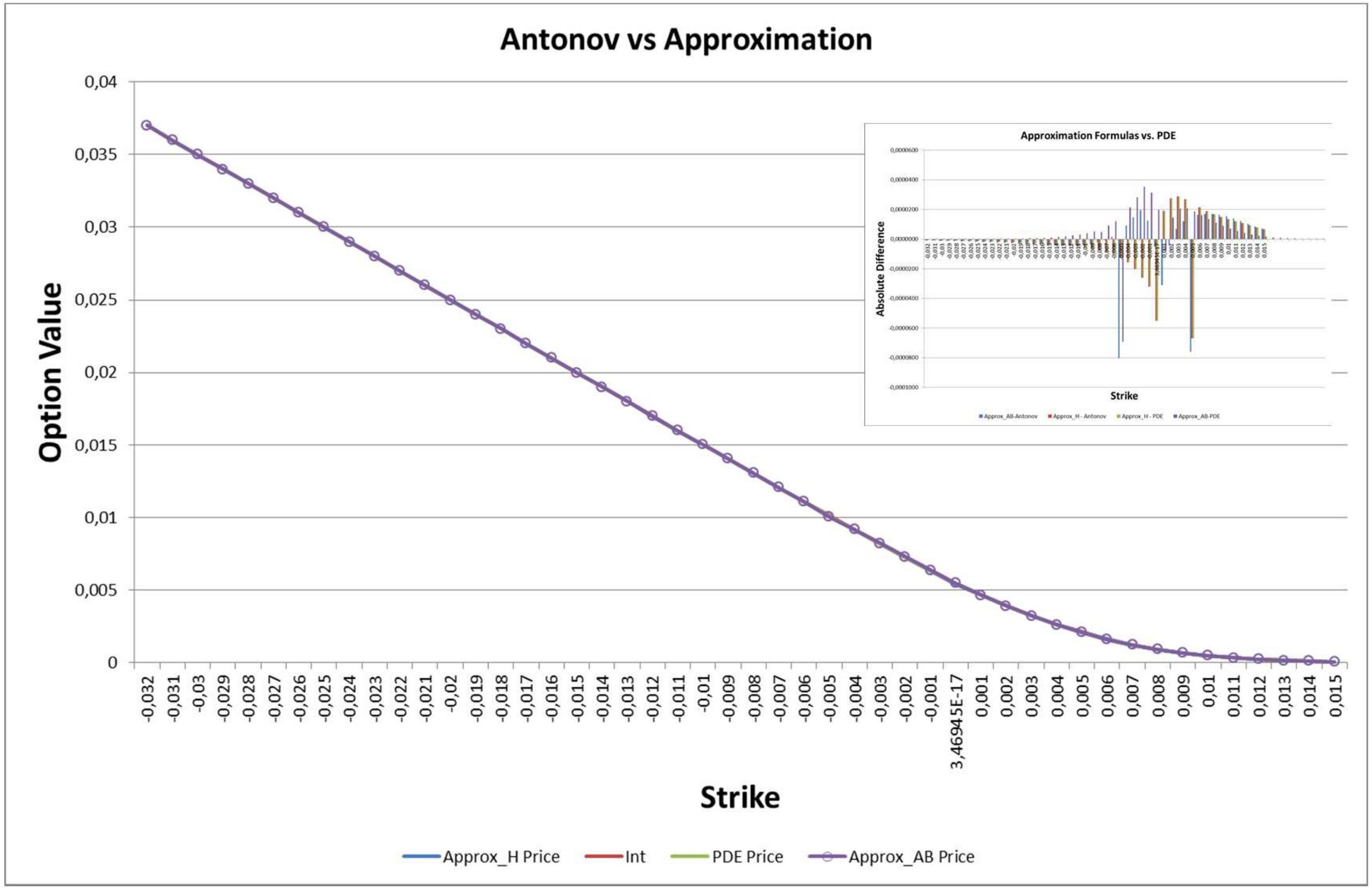
$$\begin{aligned} g_K &= -\log\left(\frac{|f|+|K|}{\xi_{KK}\sqrt{DfDK}}\right)/\xi_{KK}^2 \\ \Gamma_K &= \frac{-|K|^\beta-|f|^\beta}{1-\beta} \\ y_K &= \frac{-|K|^{1-\beta}-|f|^{1-\beta}}{1-\beta} \\ Df &= \alpha|f|^\beta \\ Dk &= \sqrt{\alpha^2+2\alpha\rho\nu y_K+\nu^2y_K^2}|K|^\beta \\ \xi_K &= \frac{\nu}{\alpha(1-\beta)}(|f|^{1-\beta}+|K|^{1-\beta}) \\ \xi_{KK} &= \frac{\log(\sqrt{1-2\rho\xi_K+\xi_K^2}-\rho+\xi_K)}{\nu(1-\rho)} \end{aligned}$$

$K \geq 0$

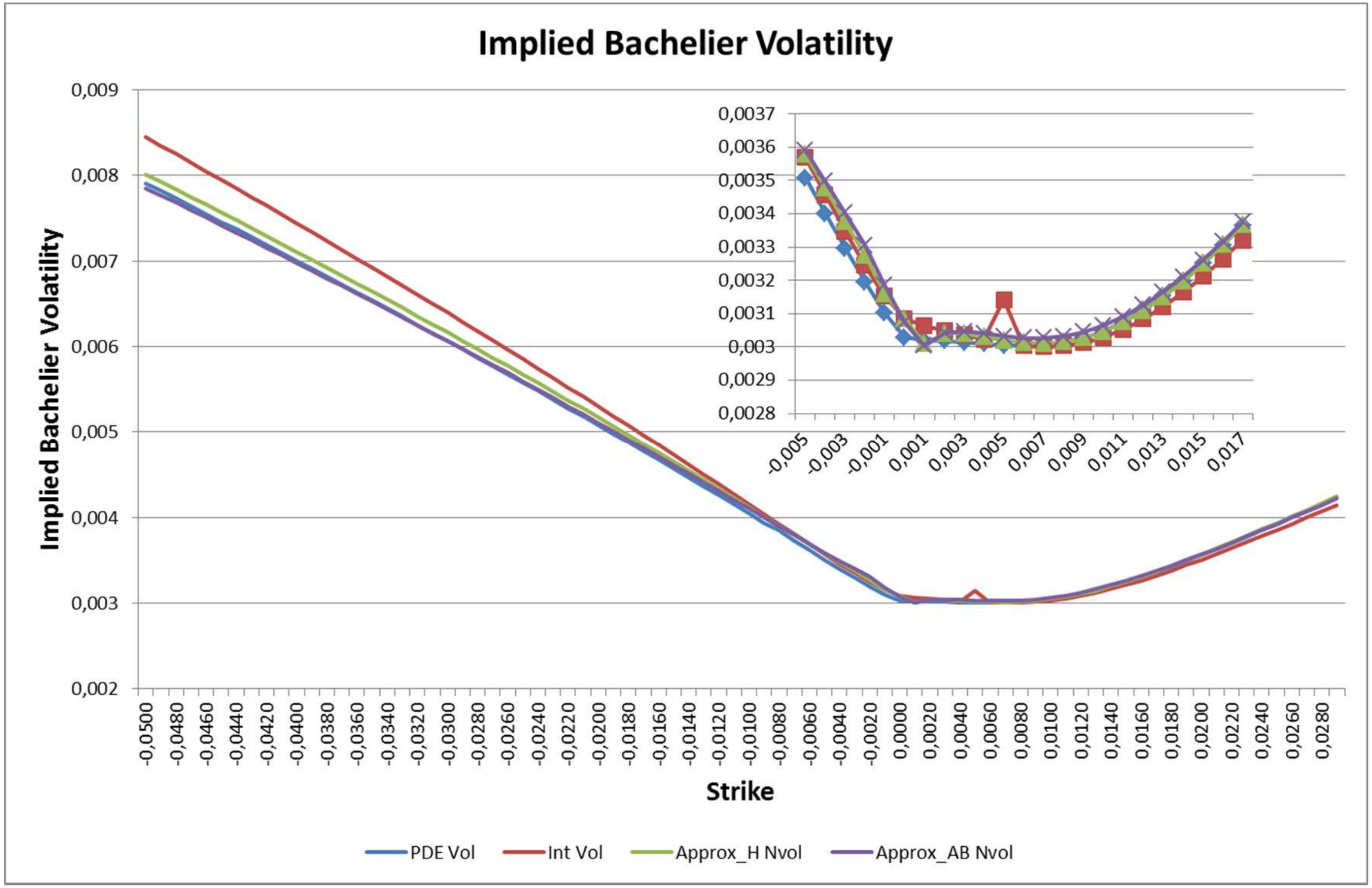
$$\begin{aligned} g_K &= -log\left(\frac{|f|-|K|}{\xi_{KK}\sqrt{DfDk}}\right)/\xi_{KK}^2 \\ \Gamma_K &= \frac{|K|^\beta-|f|^\beta}{1-\beta} \\ y_K &= \frac{|K|^{1-\beta}-|f|^{1-\beta}}{1-\beta} \\ Df &= \alpha|f|^\beta \\ Dk &= \sqrt{\alpha^2+2\alpha\rho\nu y_K+\nu^2y_K^2}|K|^\beta \\ \xi_K &= \frac{\nu}{\alpha(1-\beta)}(|f|^{1-\beta}-|K|^{1-\beta}) \\ \xi_{KK} &= \frac{\log(\sqrt{1-2\rho\xi_K+\xi_K^2}-\rho+\xi_K)}{\nu(1-\rho)} \end{aligned}$$

Free Boundary SABR - Example

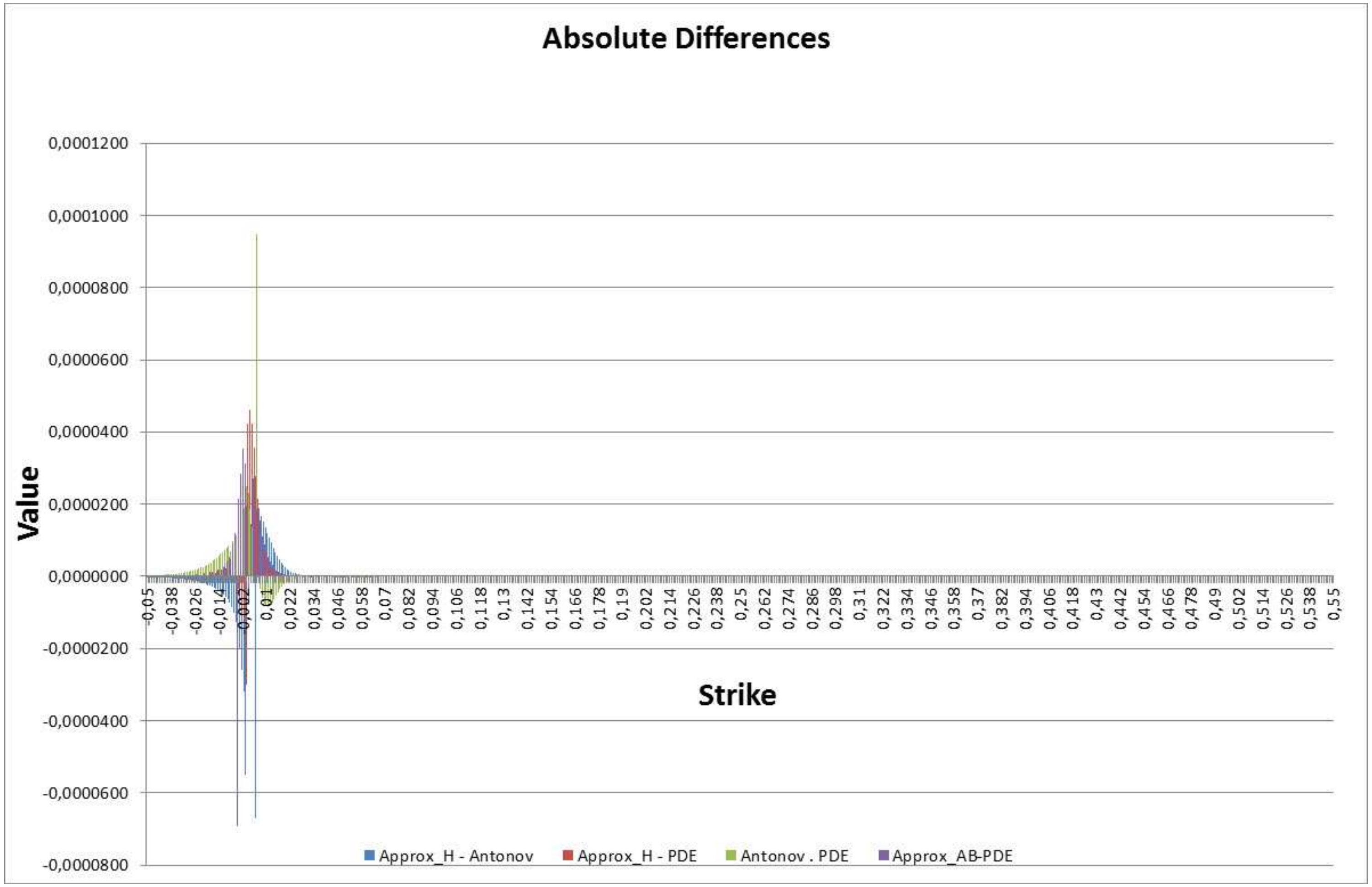
Comparison Integration, PDE, Approximation



Free Boundary SABR - Example
Comparison Integration, PDE, Approximation

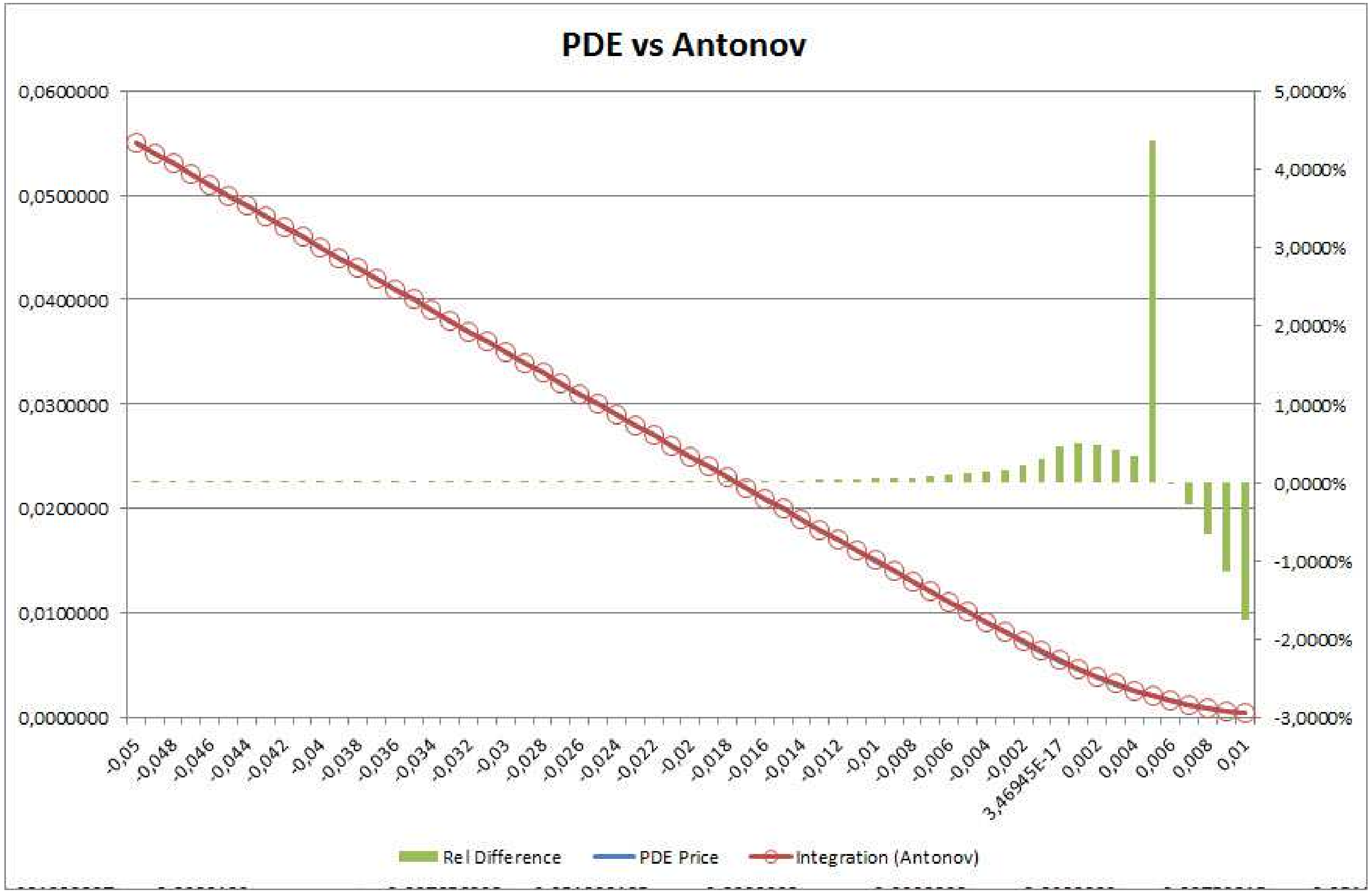


Free Boundary SABR - Example
Comparison Integration, PDE, Approximation



Free Boundary SABR - Example

Comparison Integration, PDE, Approximation



Free Boundary SABR - Example
Calculation via PDE

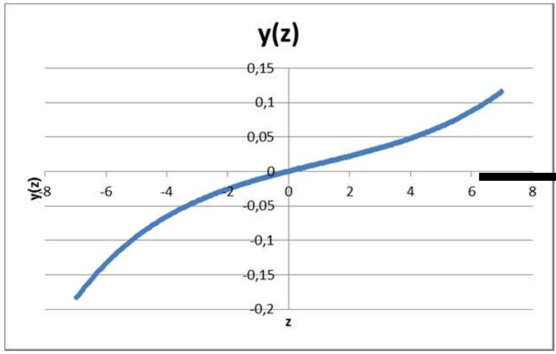
- PDE method for calculating the SABR density numerically can be adapted to the Free SABR case
- Non-Standard grids and other tricks speed up the calculation again
- Density using the PDE and option prices via numerical integration
- Being able to check the integration and the (new) approximation formulas

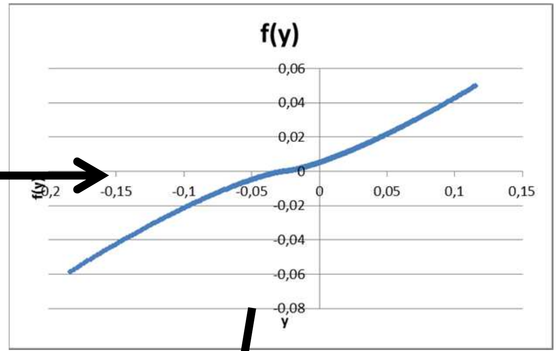
Free Boundary SABR - Example

Calculation via PDE

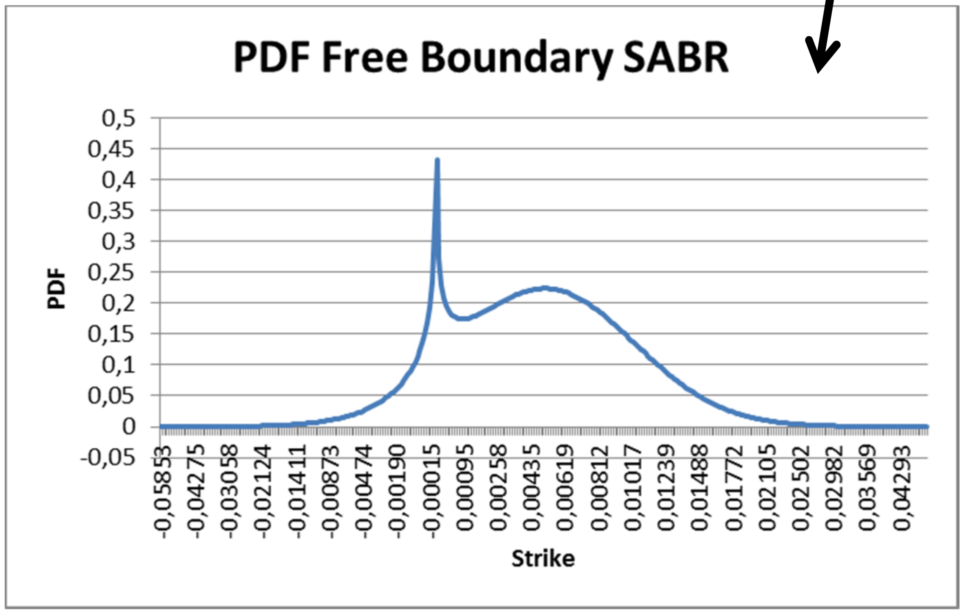
Transformation of Variables

alpha	0,011281809
beta	0,25
nu	0,3
rho	-0,3
forward	0,005
timemat	3
Nsteps	320
Tsteps	320
nd	4





Solving the PDE

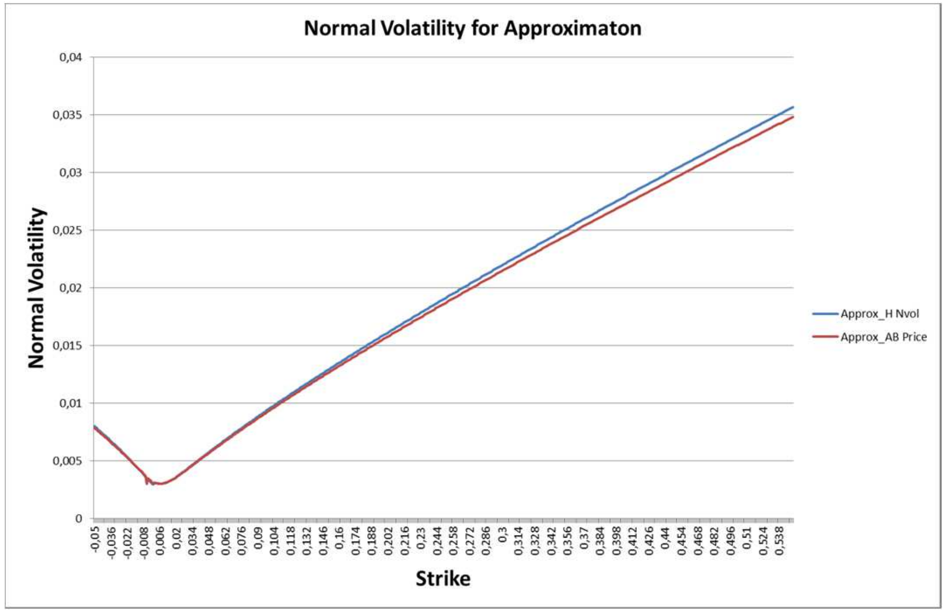


Free Boundary SABR – (Antonov et al.)
SABR gets technical

- Integrals solution for correlation 0 case and Markovian projection for other cases (might be instable for large values of absolute correlation)
- PDE approach as shown for No-Arbitrage SABR
- Approximation formulas as shown for No-Arbitrage SABR

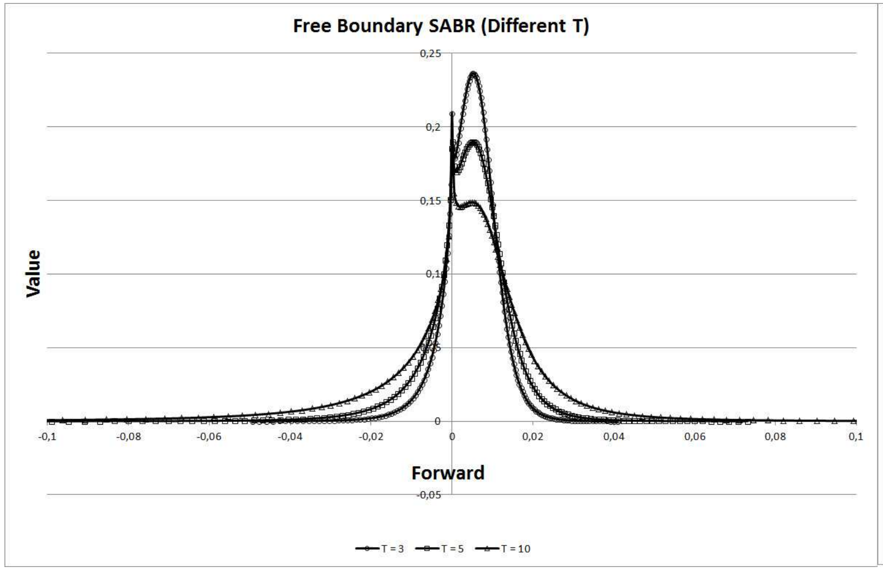
This allows to test approximation formulas and fastens the calibration.

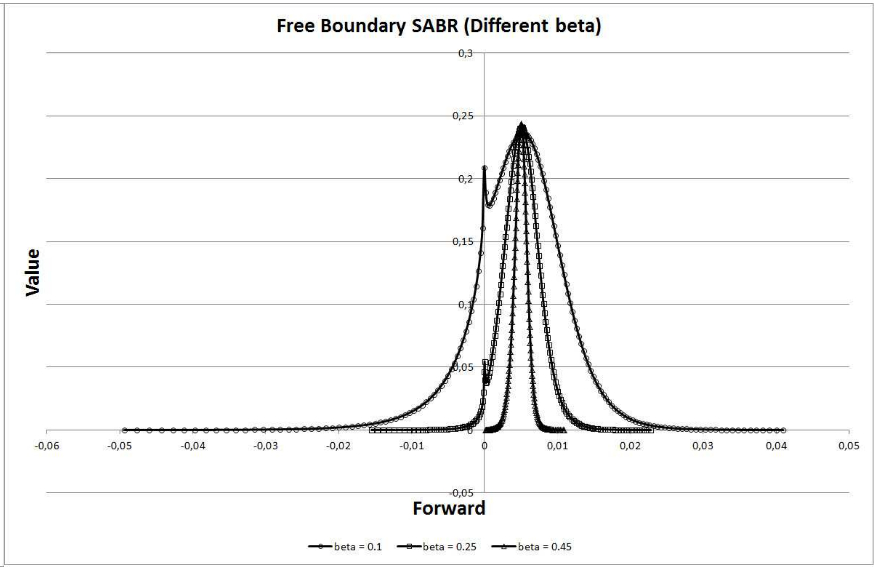
Are the good old times back?



Free Boundary SABR – (Antonov et al.)
Examples

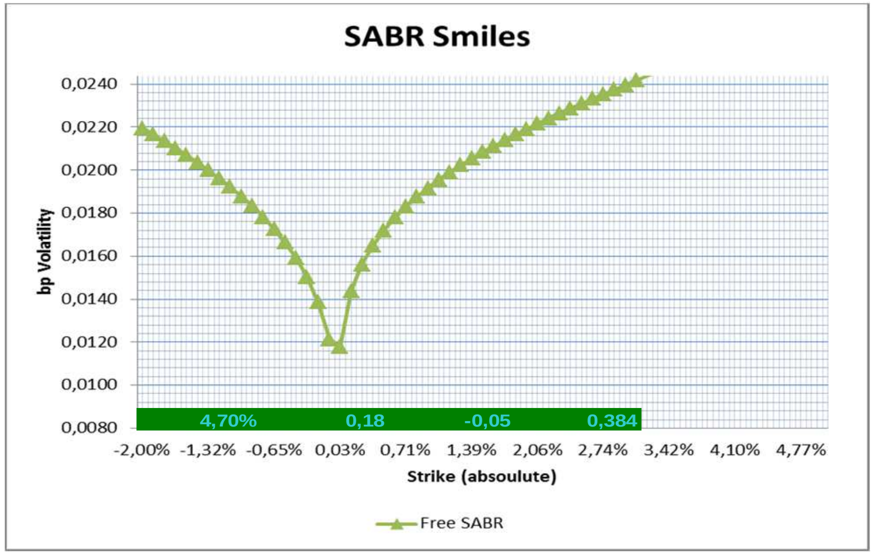
We have applied the modified PDE solution to different parameter sets. The following illustrations are for different values of T and beta.

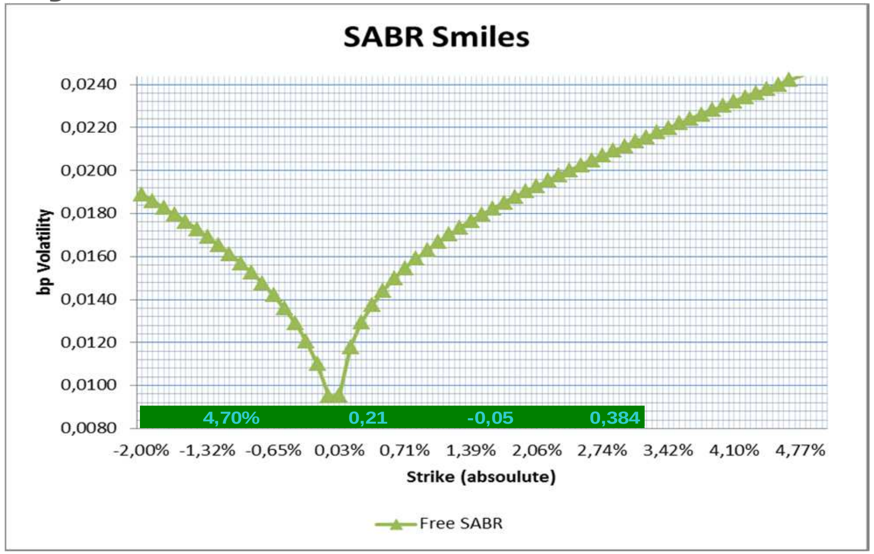


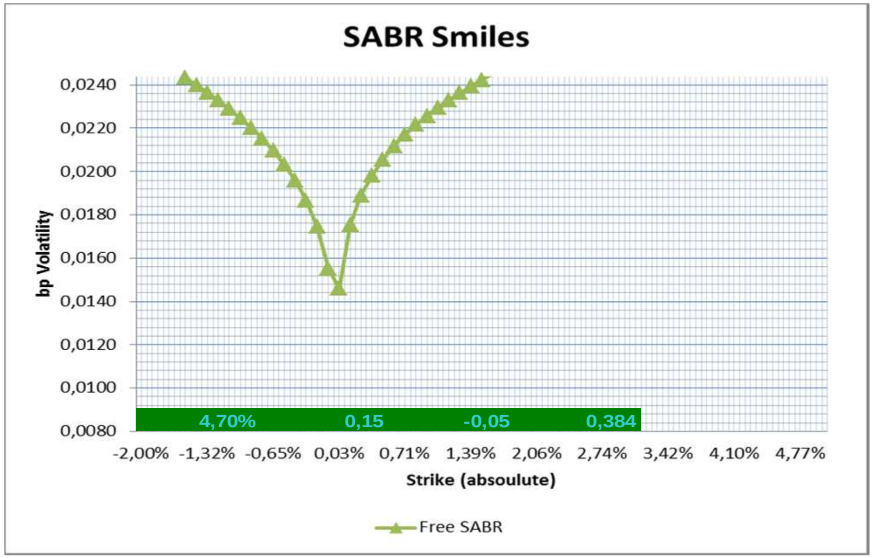


Free SABR

Bachelier Implied Volatility - beta

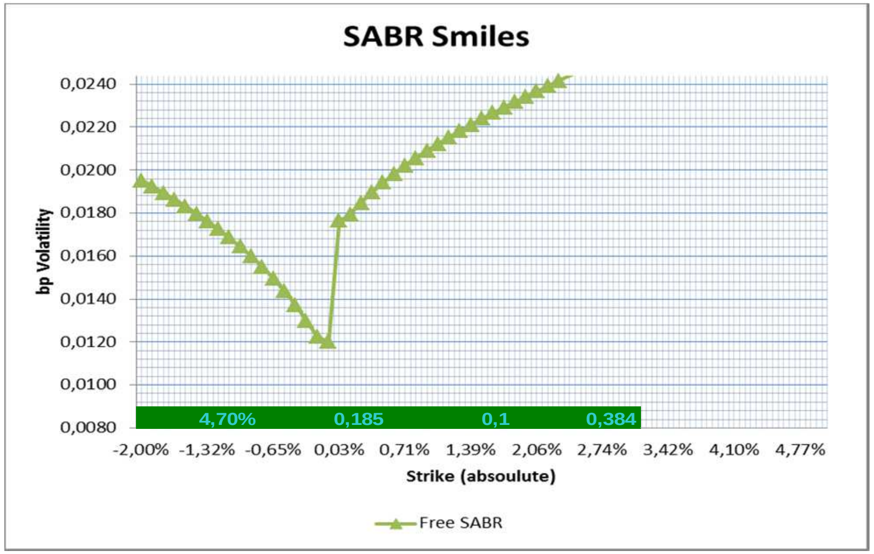


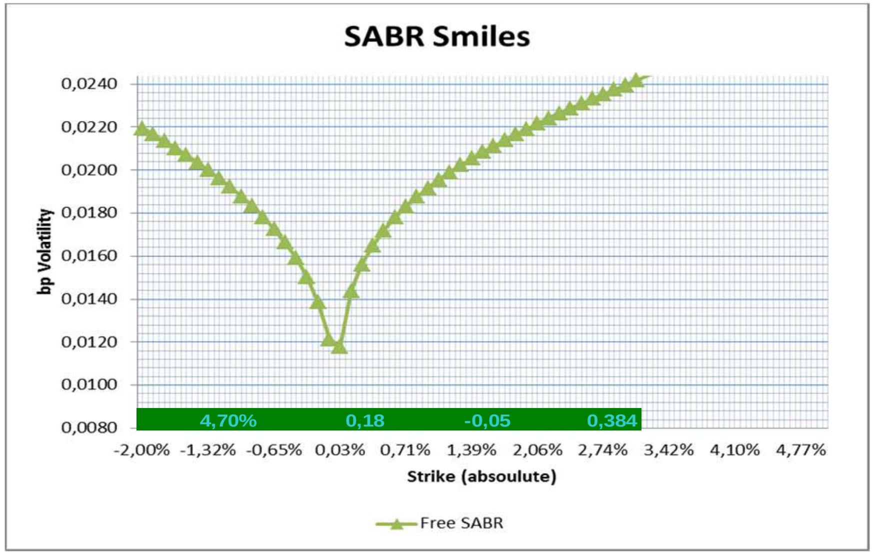


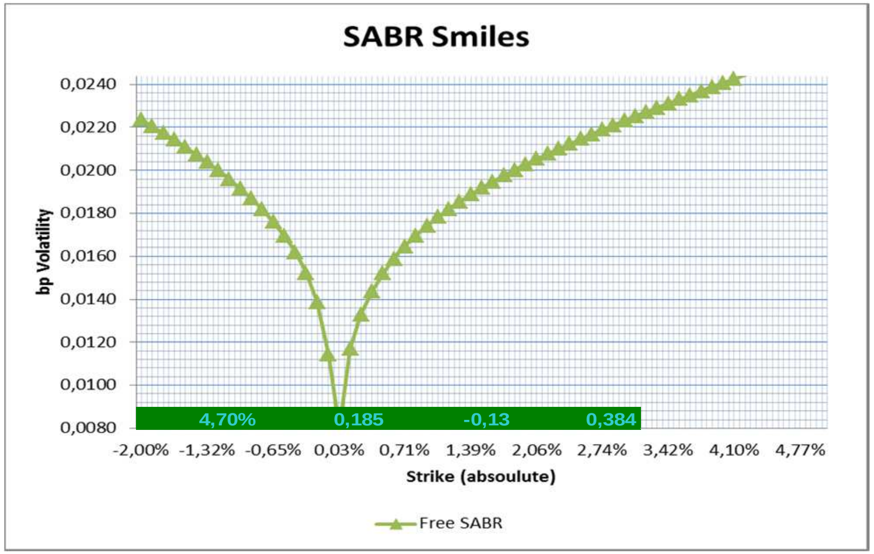


Free SABR

Bachelier Implied Volatility - rho

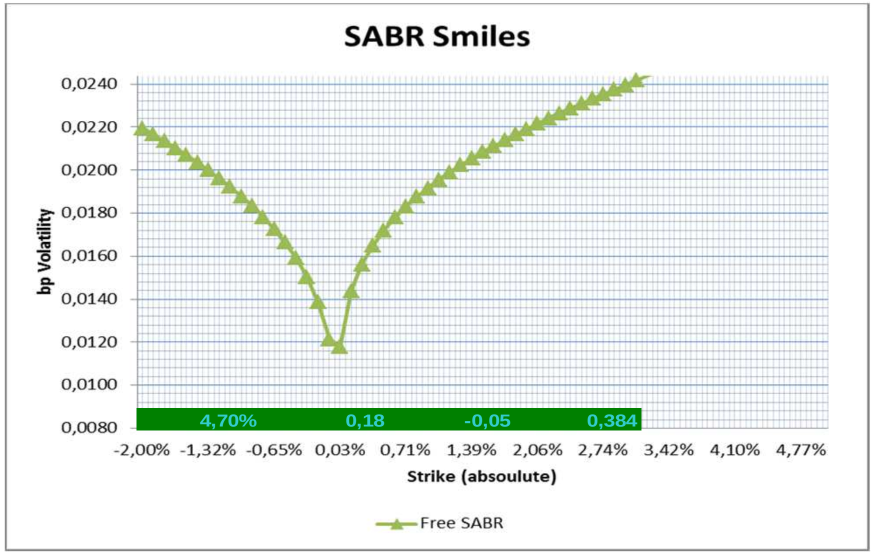


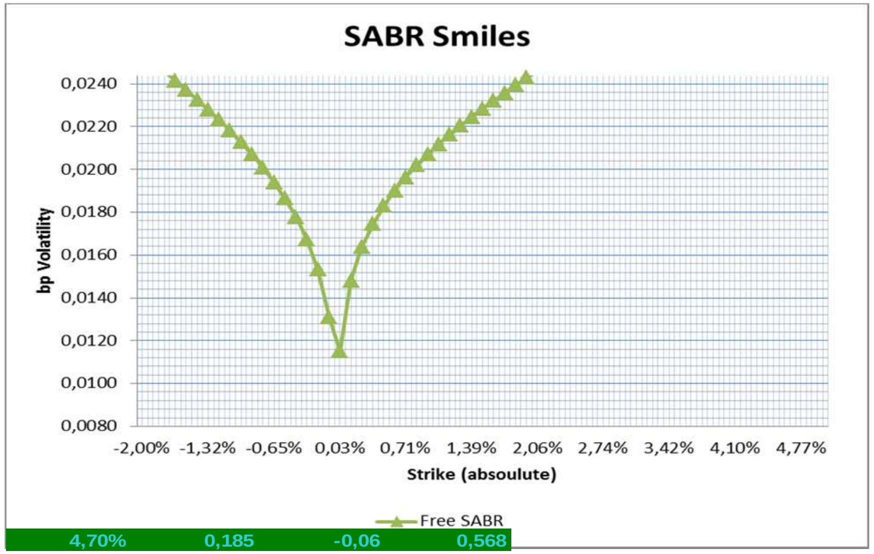


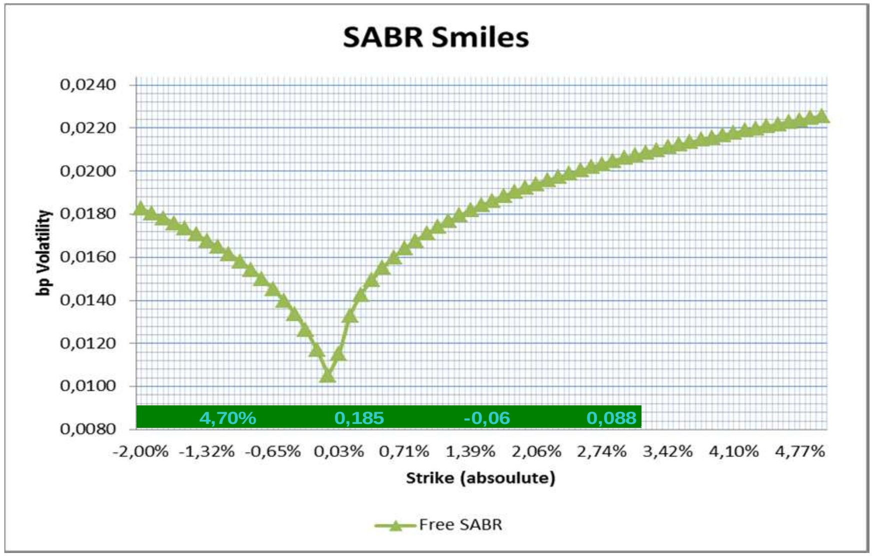


Free SABR

Bachelier Implied Volatility - nu

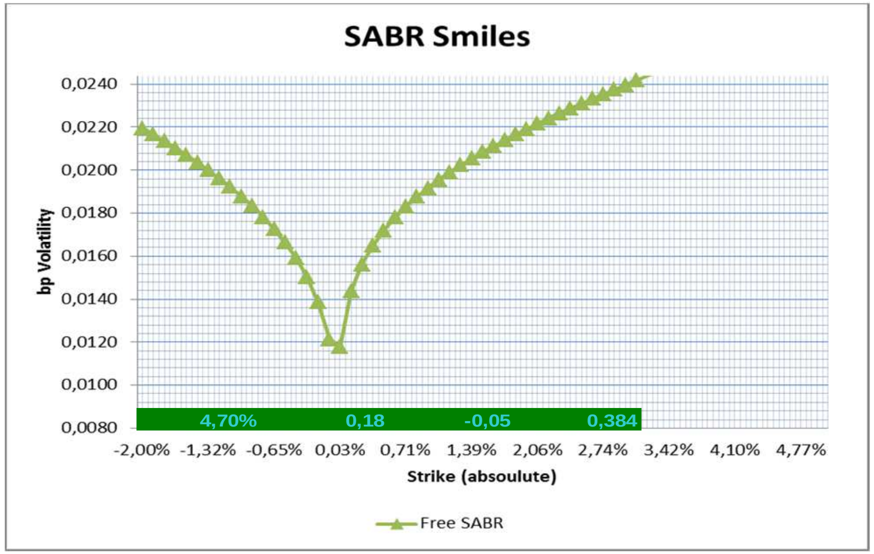


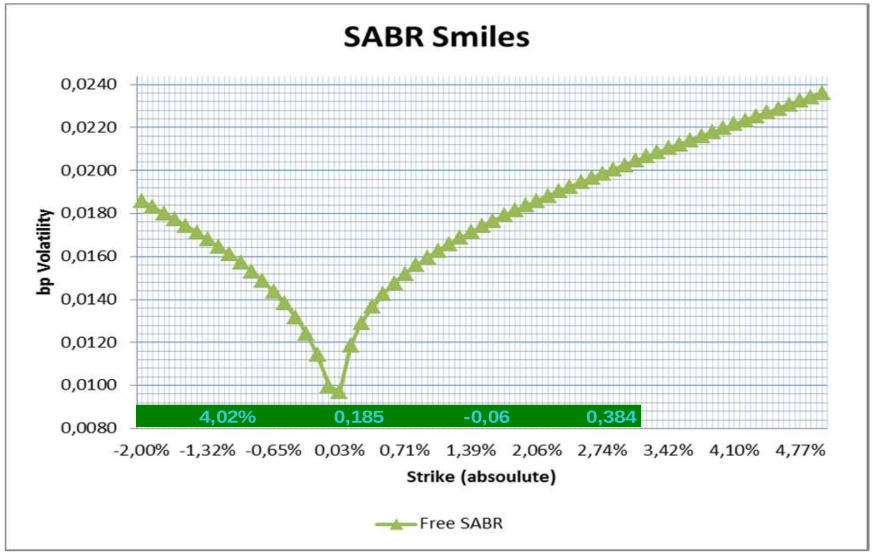


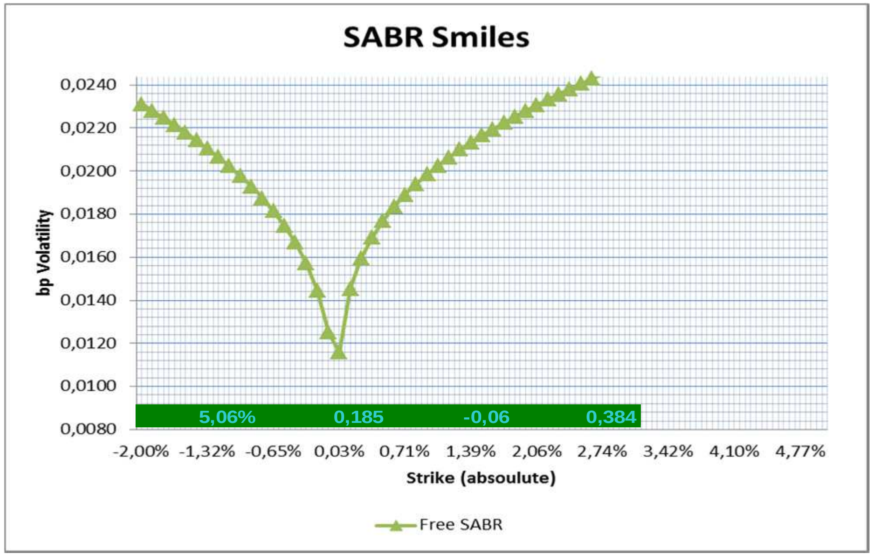


Free SABR

Bachelier Implied Volatility - alpha



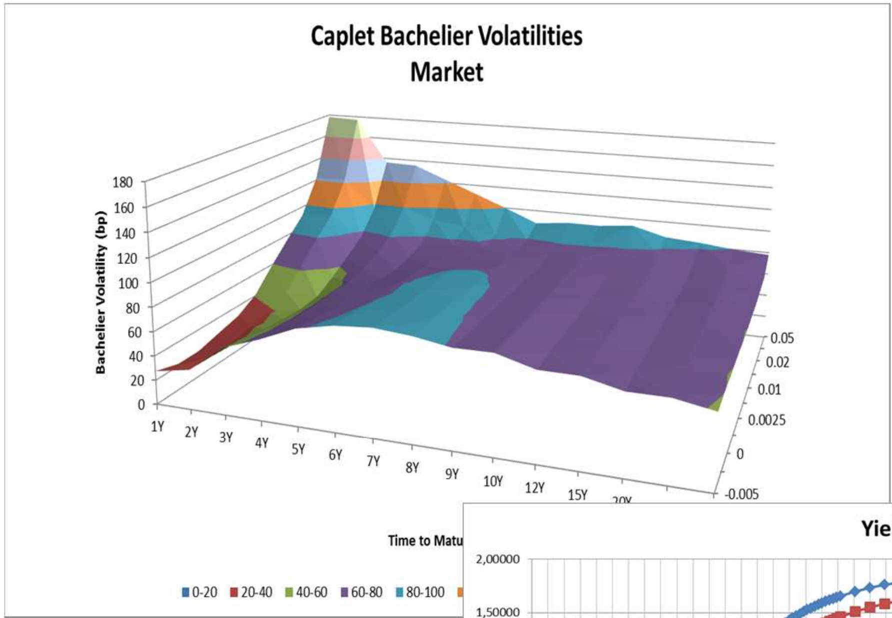


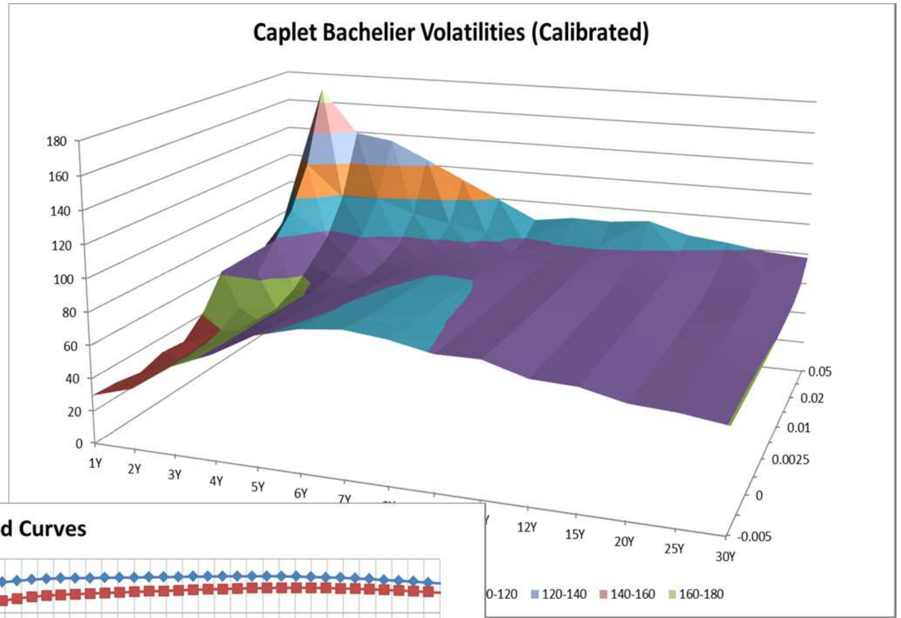


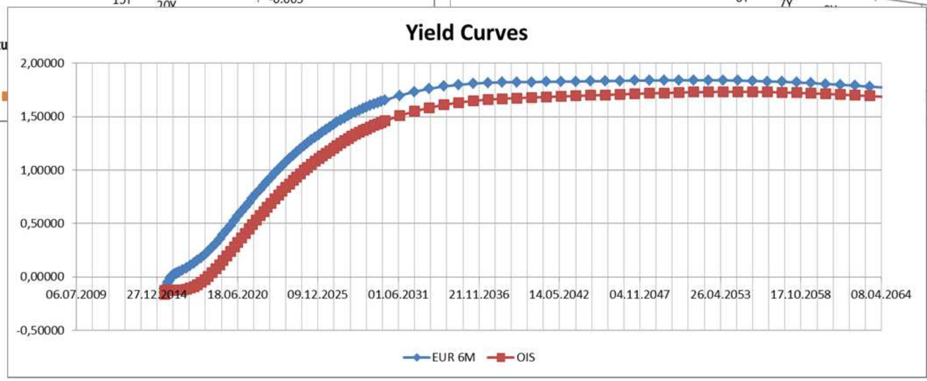
Free Boundary SABR – (Antonov et al.)

Calibrating Caps with Free Boundary SABR

Calibration example on EUR June data.

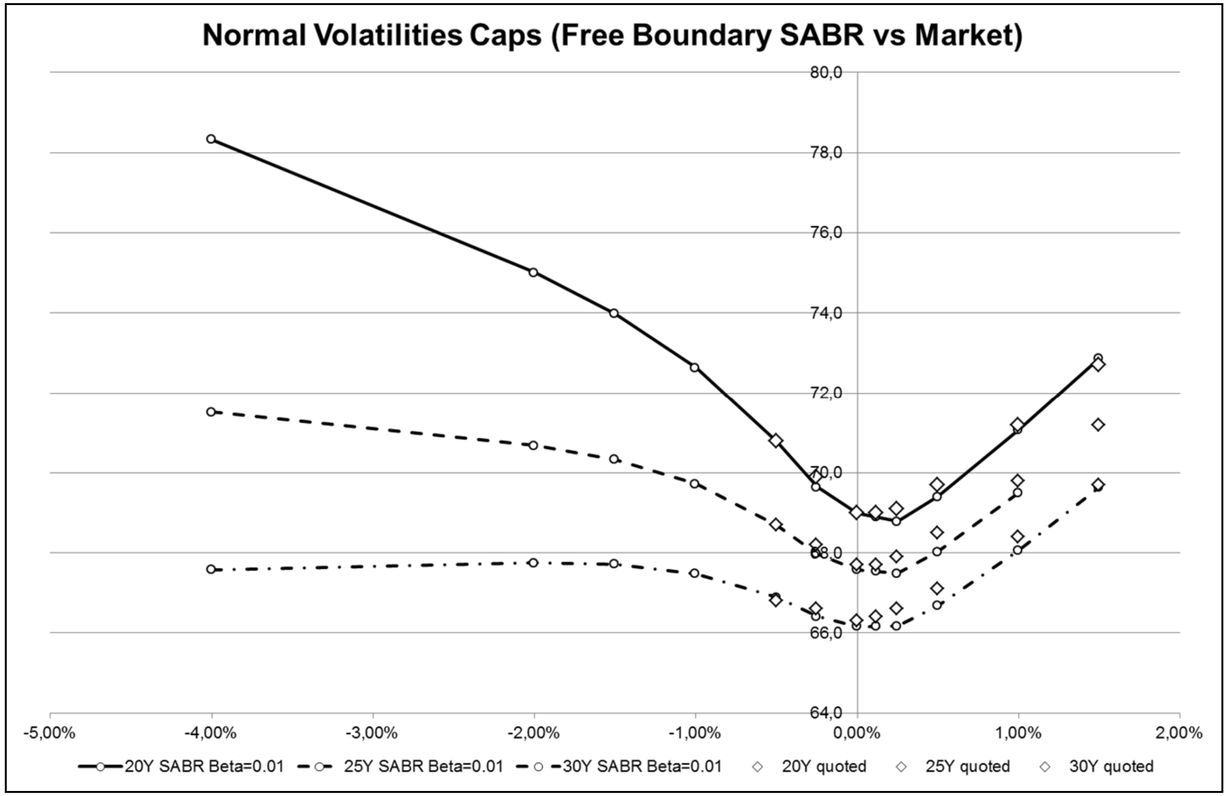




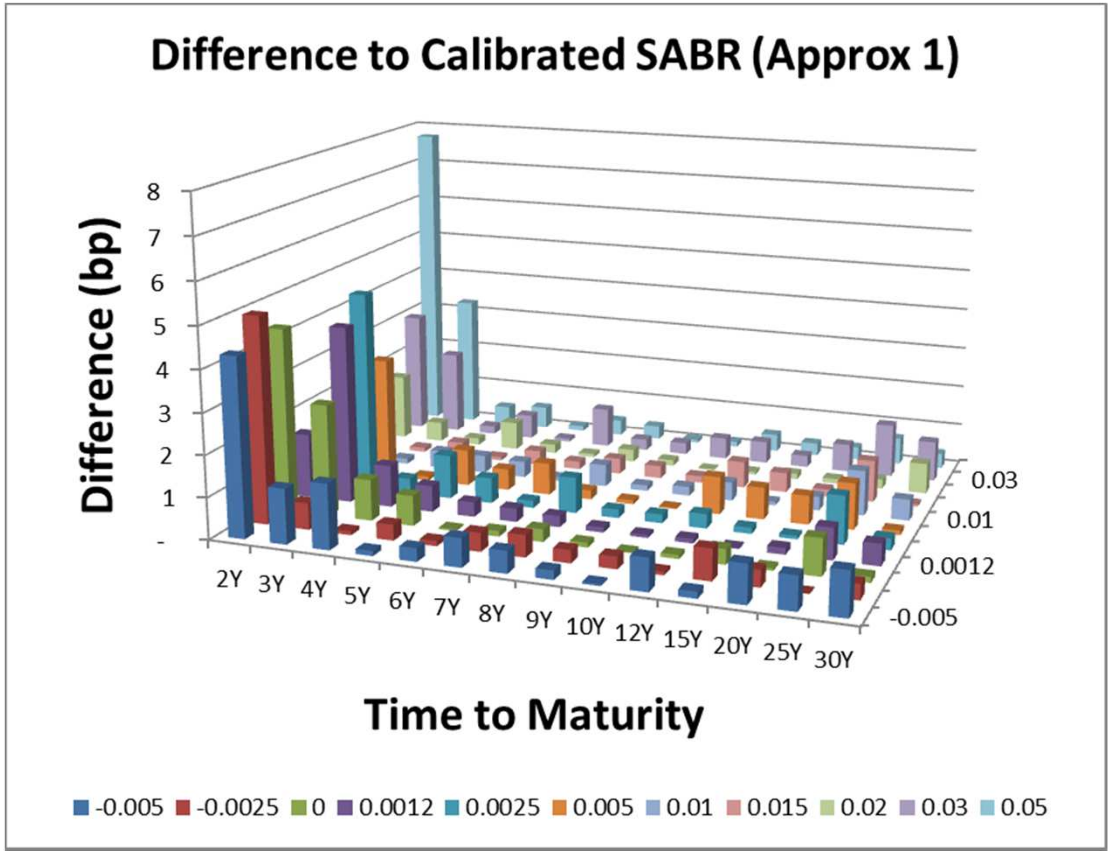


Free Boundary SABR – (Antonov et al.)

Calibrating Caps with Free Boundary SABR



Free Boundary SABR – (Antonov et al.)
Calibrating Caps with Free Boundary SABR



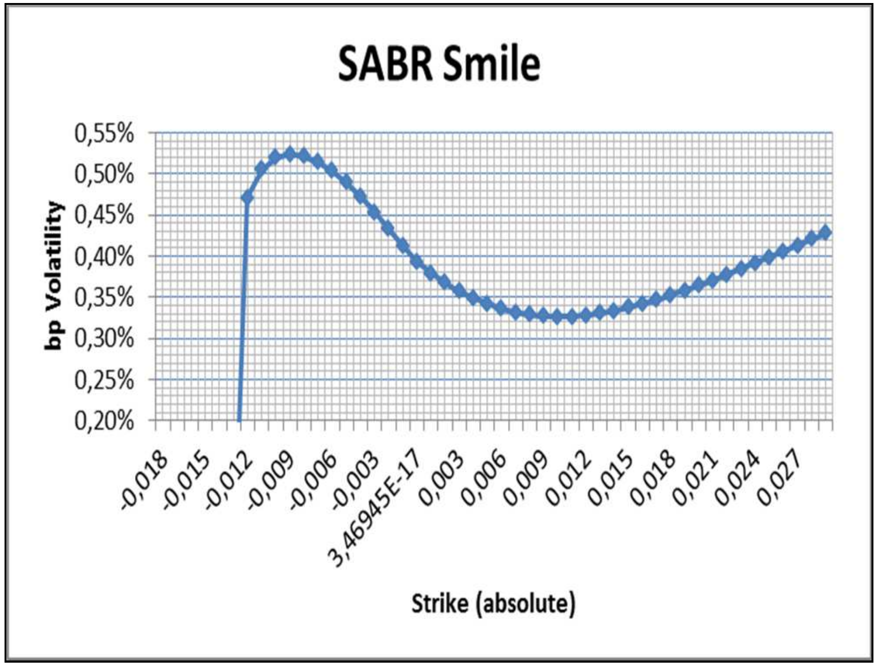
SABR
Shape of the (Bachelier) Smile

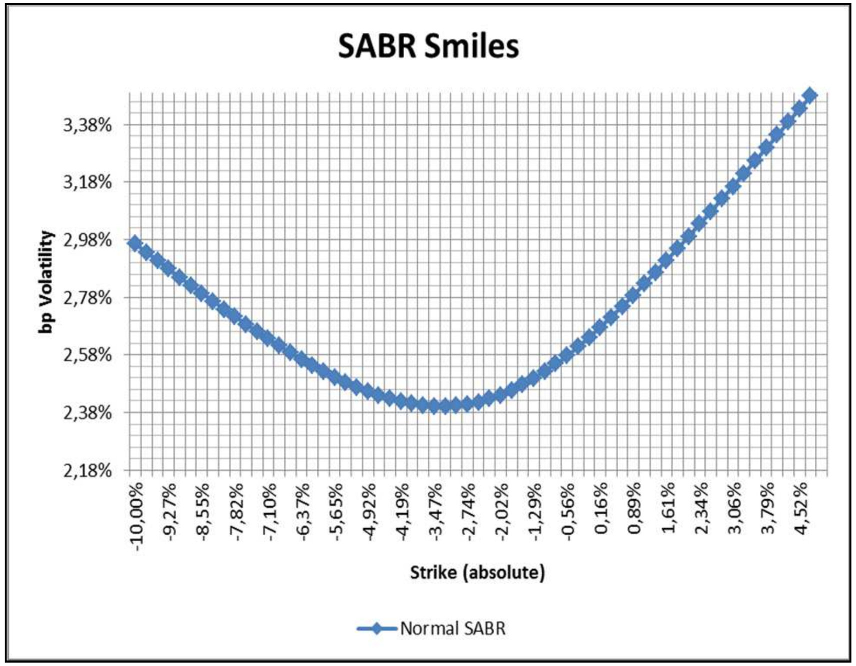
DD SABR / No Arb SABR

Implied Volatilities tend decrease for small strikes (‚SABR Knee‘)

Free SABR / Normal SABR

Implied Volatilities do not decrease for small strikes





SABR – Sticky Absorbing SABR

A New Parametrization

The SABR model was:

$$\begin{aligned} dF(t) &= v(t)C(F(t))dW_1(t) \\ dv(t) &= \gamma v(t)dW_2(t) \\ \langle dW_1(t), dW_2(t)\rangle &= \rho dt \\ F(0) &= f \\ v(0) &= v_0 \end{aligned}$$

To be able to handle negative rates we may choose:

$$C(F) = (F+a)^\beta$$ **Displaced SABR**

$$C(F) = 1$$ **Normal SABR**

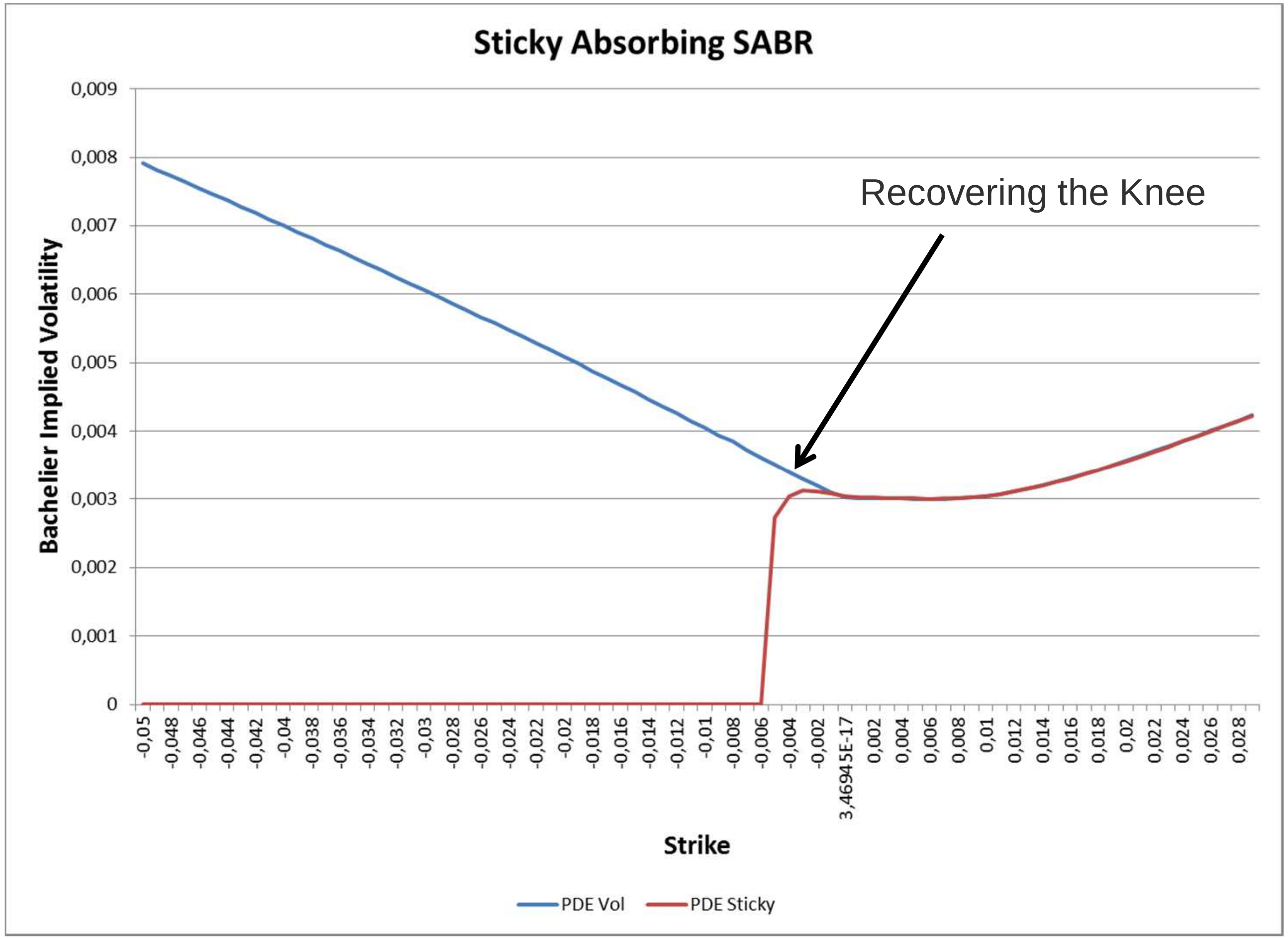
$$C(F) = |F|^\beta$$ **Free SABR**

$$\begin{cases} C(F) = |F+a|^\beta \\ F_{\text{Min}} = -b \end{cases}$$

Sticky Absorbing SABR

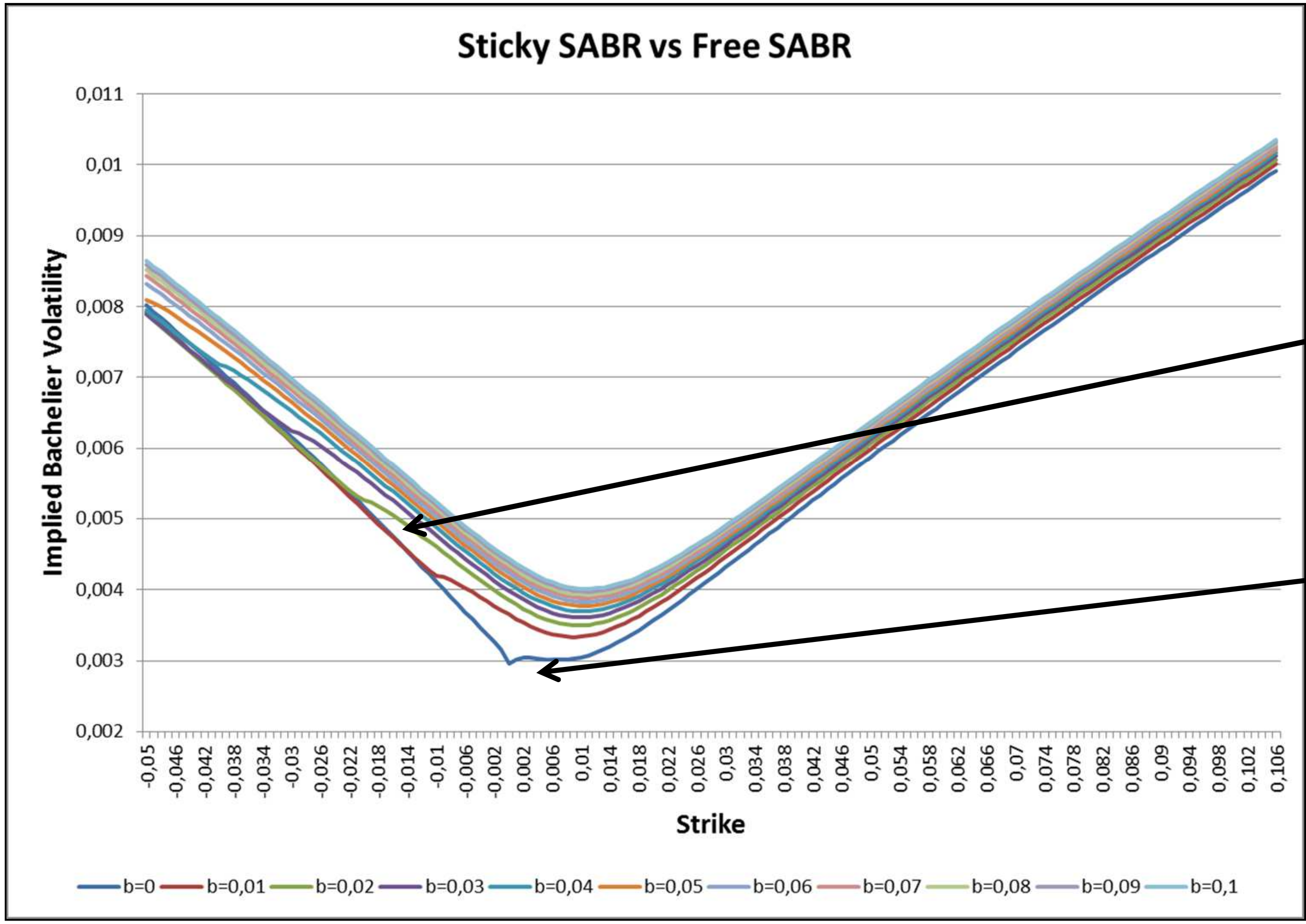
SABR – Sticky Absorbing SABR

Recovering the SABR Knee (using $F_{\mathrm{Min}} = -b$)



SABR – Sticky Absorbing SABR

Controlling the Stickyness of Rates (using $C(F) = |F+a|^{\beta}$)



Sticky SABR

Moving Stickyness around

Free SABR

SABR
Summary

- Using new numerical methods the SABR model can be tackled
- Integration, PDE and Approximation formulas are available
- The probability density and prices can be calculated efficiently
- The Free SABR model, the Displaced and the Normal SABR can accomodate negative rates
- The Free SABR and Normal SABR imply that the implied volatilities do not tend to 0 for very small number
- To obtain this result DD SABR or an artifical lower bound should be applied

Mixture of Models

- ZABR, SABR -

Mixture of Models

Rationale – Controlling the Wings

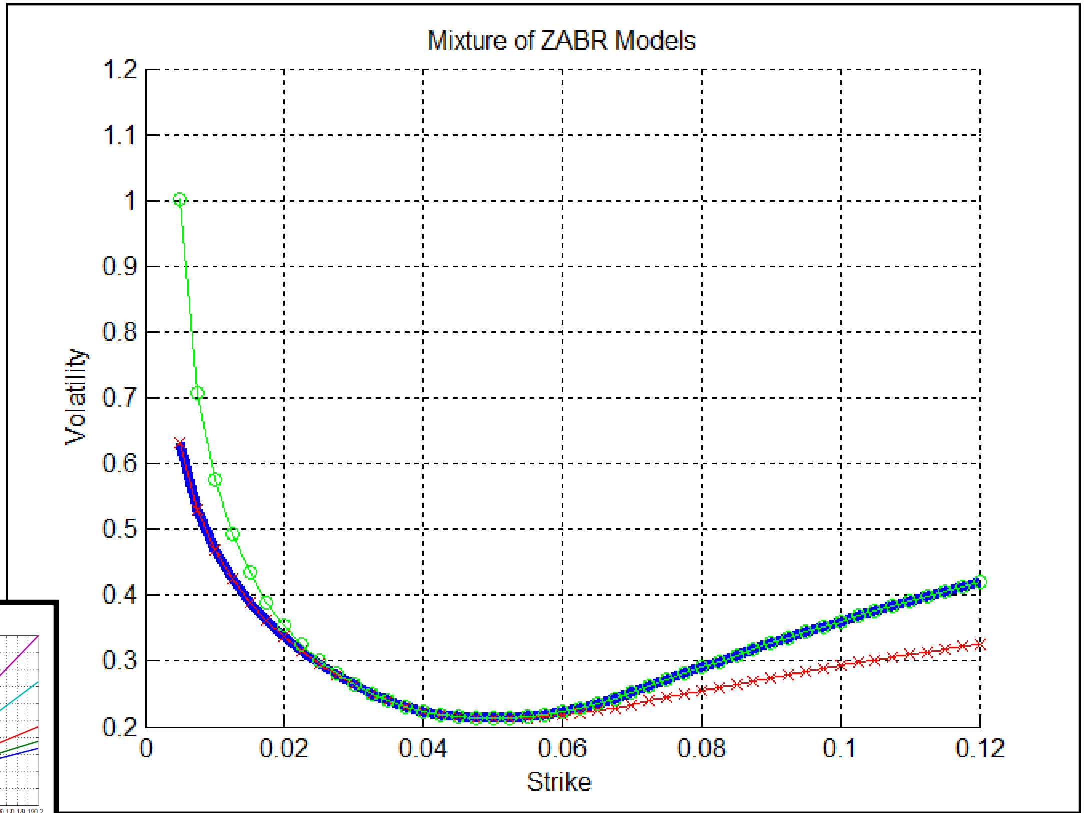
- Use different models / different parametrizations which are analytically tractable
- Take a convex combination
- This was first done with Shifted Log-Normal models (different displacements)
- We can apply the ZABR or the (Free Boundary) SABR model
- This methodology is for pricing Vanillas since it does not give a reasonable model dynamics -> See Piterbarg „Mixture of Models: A Simple Recipe for a … hangover“ (2009)

ZABR – Mixture of ZABR Models
Smile Control by Mixing?

Mixing ZABR models to control the behaviour of the wings.

This allows to control the behaviour of the left or the right wing separately.

In contrast to simply using ZABR.



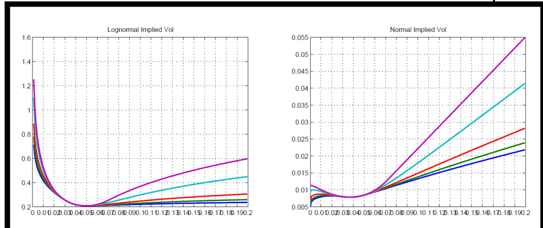
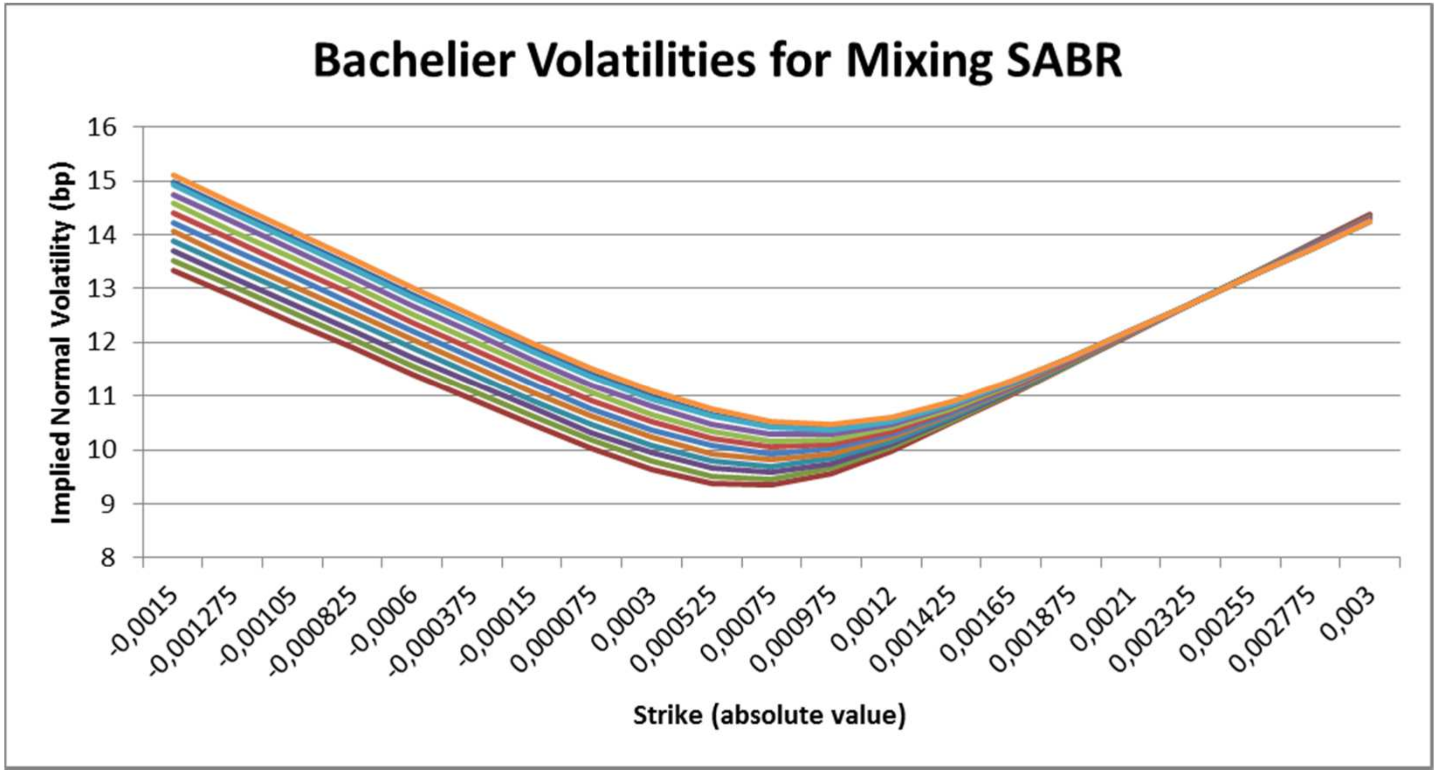


Figure : ZABR Implied log-normal (left) and normal volatilities (right) for $\beta_v = 0, 0.5, 1, 1.5, 1.7$ and the parameters from $v_0 = 0.0873$, $\beta = 0.7$, $\rho = -0.48$, $\gamma = 0.47$, $f = 0.0325$ and $T = 10$.

SABR – Mixture of Different SABR Models

Smile Control by Mixing – Antonov Approach



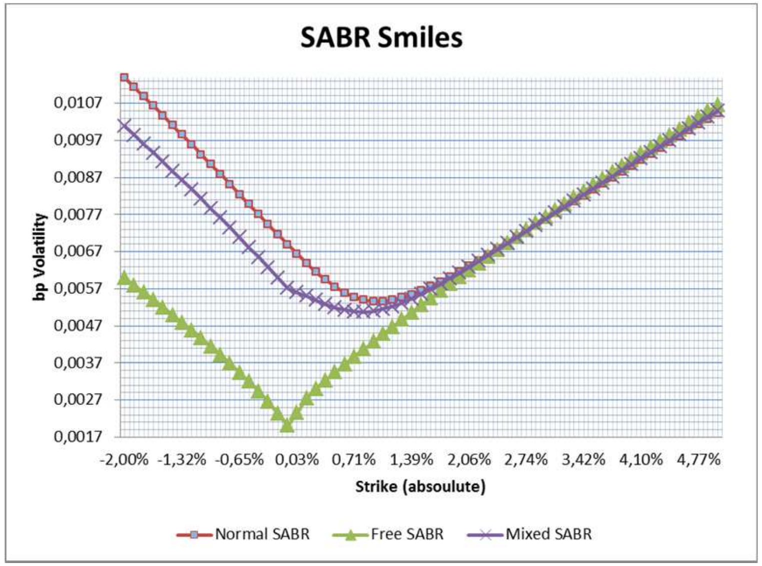
The same mixing approach can be applied to SABR models.

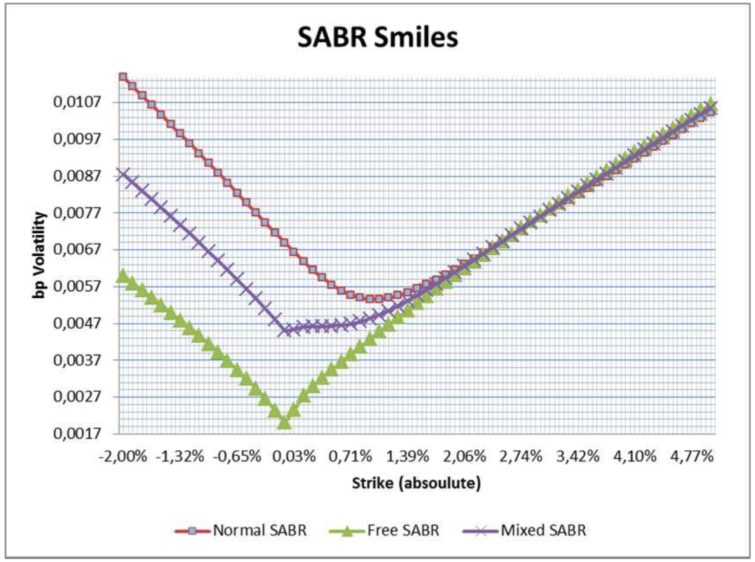
In this case Antonov et al. suggest to mix free SABR with zero correlation SABR.

However, using their approach prices have to be calculated and transformed to implied Bachelier volatilties. The approach recently put forward by Kienitz only considers implied Bachelier volatilties.

SABR – Mixture of Different SABR Models

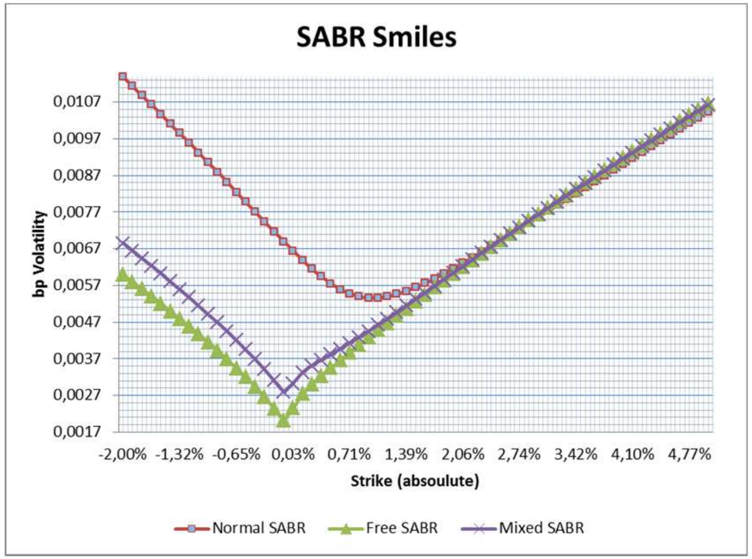
Smile Control by Mixing – Kienitz Approach





Here we consider the mixing of SABR models while keeping the tail behaviour to the right the same for both SABR models.

We use the approximation formula in the Free SABR context.



Negative Rates, SABR, CMS - Literature

Paper, Books

Papers

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