Overview
Negative Rates, SABR PDE and Approximation

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Negative Rates
- Markets, Models, Numerics -
The good old times - Markets
Interest rate World

• Rates are significantly positive

• Volatilities are at „normal“ levels

• Quotes are in log-normal volatility or premium

• There was a simple to code approach for SABR to a model which gives a very well fit of the market observed volatility structures

\[
\begin{align*}
dS(t) &= S\sigma dW(t), \quad S(0) = s_0 \\
C_{LN}(S(0), K, T, \sigma) &= S(0)N(d_1) - KN(d_2) \\
P_{LN}(S(0), K, T, \sigma) &= KN(-d_2) - S(0)N(-d_1)
\end{align*}
\]
The (popular) SABR model is specified by the following system of Stochastic Differential Equations:

\[
\begin{align*}
    dF(t) &= \nu(t)C(F(t))dW_1(t) \\
    d\nu(t) &= \gamma \nu(t) dW_2(t) \\
    \langle dW_1(t), dW_2(t) \rangle &= \rho dt \\
    F(0) &= f \\
    \nu(0) &= \nu_0
\end{align*}
\]

The function $C$ is the local volatility. For the standard SABR we have $C(F) := F^\beta$. 
SABR - Approximation Formula
Black Formula with SABR Volatility

An approximation formula for Log-normal (Black) volatilities in the SABR setting:

\[
\sigma_{BS}(K, T) \approx \frac{v_0}{(fK)^{\frac{1-\beta}{2}} \left(1 + \frac{(1-\beta)^2}{24} \log^2(f/K) + \frac{(1-\beta)^4}{1920} \log^4(f/K) + \ldots\right)} \left(1 + \frac{(1-\beta)^2 v_0^2}{24(fK)^{1-\beta}} + \frac{\rho \beta \gamma v_0}{4(fK)^{1-\beta}} + \frac{\gamma^2 (2 - 3 \rho^2)}{24} \right) T + \ldots \right),
\]

\[
z = \frac{\gamma}{v_0} (fK)^{\frac{1-\beta}{2}} \log(f/K), \quad x(z) = \log \left(\frac{\sqrt{1 - 2z\rho + z^2} + z - \rho}{1 - \rho}\right)
\]
SABR – Approximation Formula

General Bachelier Formula with SABR Volatility

An approximation for the Bachelier (Normal) volatilities in the SABR setting:

\[
\sigma_B = \frac{v_0(f - K)}{\int_K^f C(g)^{-1} dg} \left( \frac{\xi}{x(\xi)} \right) \\
\left( 1 + \left( G v_0^2 + \frac{\rho \gamma v_0}{4} \frac{C(f) - C(K)}{f - K} + \frac{2 - 3 \gamma^2}{24} \right) T + \ldots \right)
\]

\[
\xi := \frac{\gamma}{v_0} \int_K^f C(g)^{-1} dg, \quad x(\xi) := \log \left( \frac{\sqrt{1 - 2 \rho \xi^2} - \rho + \xi}{1 - \rho} \right)
\]

\[
G := \log \left( \frac{\int_K^f \frac{C(f)C(K)}{C(g)} dg}{\int_K^f \frac{C(g)^{-1} dg}{f - K}} \right) / \left( \int_K^f C(g)^{-1} dg \right)^2
\]
SABR – The SABR Parameters

\[ \beta \]

\[ \nu_0 \]

\[ \rho \]

\[ \gamma \]
SABR – Die SABR Parameter (Implied Bachelier)
SABR - Calibration

Input

VOLAS

FORWARDS
SABR - Calibration
Output

Usually $\beta$ fix or calibrated using CMS quotes

SABR Parameters

Root Mean Square

Number of Iterations

Differences to Market
SABR - Calibration

Anything to keep an eye on?

- „Smart Parameters“ (Nice starting values for local solvers) help to stabilize your SABR calibration and reduces the number of iterations which leverages the calibration speed.
- The ATM should be fit perfectly
- Calibration to all EUR Swaption < 0.5 sec.
  (even in VBA)
SABR
The age of low or even negative rates
Market Quotes

Caps and Floors - ICAP

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<th>ATM</th>
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<td>30Y</td>
<td>1.29</td>
<td>70.0</td>
</tr>
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</table>

(Very) high volatilities

Negative strikes
After August 2007 several market practices and usances changed significantly.

Market parameters such as forward swap rates became very small and the corresponding volatilities became very high.

OIS Discounting and credit/liquidity issues and appearance of multi tenored curves

We consider the impact on the standard SABR model
Negative Rates
Which models can now be used?

- We have to handle curves
  - Generally the negative rates do not have that impact on curves construction and is a standard process these days
  - The appearance of significant basis spreads have much more impact

- We still need to price options
  - Caps/Floors
  - Swaptions
  - CMS Caps/Floors
  - CMS Spread Options
Negative Rates – Models
Pricing Options with negative Strikes

- The displaced diffusion or shifted log-normal model
  \[ dS(t) = (S + b)\sigma dW(t), \quad S(0) = s_0 \]
  \[ C_{DD}(S(0), K, T, \sigma) = (S(0) + b)N(d_1) - (K + b)N(d_2) \]
  \[ P_{DD}(S(0), K, T, \sigma) = (K + b)N(-d_2) - (S(0) + b)N(-d_1) \]

- The Bachelier (or Normal) model
  \[ dS(t) = \sigma_N dW(t), \quad S(0) = s_0 \]
  \[ C_{N}(S(0), K, T, \sigma) = (S(0) - K)N(d(\sigma_N)) + \sigma\sqrt{T}n(d(\sigma_N)) \]
  \[ P_{N}(S(0), K, T, \sigma) = (K - S(0))N(-d(\sigma_N)) + \sigma\sqrt{T}n(d(\sigma_N)) \]

As we have already seen there are market quotes
Negative Rates – Models
Pricing Options with negative Strikes

• The displaced diffusion or shifted log-normal model

To derive the implied volatility of Black-Scholes or Shifted-LogNormal type the implementation from „Let’s be rational“ by Peter Jäckel can be used. A reference implementation is provided (www.jaeckel.org)

• The Bachelier (or Normal) model

To derive the implied volatility of Bachelier type the implementation form „Fast and Accurate Analytic Basis Point Volatility“ by Fabien Le Floc’h can be used. A reference implementation is provided (http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2420757)
Negative Rates - Markets
Implications for a banks business

- Zero Strike Floors (Implicit in many bonds)
- Options with negative strikes
- Model Choice (Construction of Volatility Surfaces/Hedges/Exotics/...)
- IT Systems (Implementation of new models, adjusting existing models)
- Regulations
SABR, No-Arb SABR, Free Boundary SABR
- SDE, Parameters, Numerics -
SABR – Negative Rates
Choosing Parametrizations

Taking again the SABR model:

\[
\begin{align*}
    dF(t) &= \nu(t)C(F(t))dW_1(t) \\
    d\nu(t) &= \gamma \nu(t)dW_2(t) \\
    \langle dW_1(t), dW_2(t) \rangle &= \rho dt \\
    F(0) &= f \\
    \nu(0) &= \nu_0
\end{align*}
\]

To be able to handle negative rates we may choose:

\[
\begin{align*}
    C(F) &= (F + a)^\beta & \text{Displaced SABR} \\
    C(F) &= 1 & \text{Normal SABR} \\
    C(F) &= |F|^\beta & \text{Free SABR}
\end{align*}
\]
To be able to use the SABR model in a negative rates setting we have to make sure that:

- Efficient and Fast Pricing is possible (e.g. for calibration)
- Monte Carlo Methods for pricing are available (e.g. when combined with a term structure model or a hybrid model)

What are the numerical techniques we have to apply?

- PDE
- Monte Carlo
- Approximation
- Integration
SABR – Integration Formulas (Antonov et al.)
2D Integration 2012 but zero correlation

\[
C(K, T) = (f - K)^+ \\
+ \frac{2}{\pi} \sqrt{Kf} \left( \int_{lb}^{ub} \frac{\sin(\nu|\Phi(s)|)}{\sinh(s)} \frac{G(\gamma^2 t, s)}{\sinh(s)} ds \right) \\
+ \sin(\nu|\pi|) \int_{lb}^{+\infty} \frac{e^{-\nu|\Psi(s)|}}{\sinh(s)} G(\gamma^2 t, s) ds
\]

\[
lb = \text{arcsinh} \left( \frac{\gamma|q_K - q_0|}{v_0} \right), \quad ub = \text{arcsinh} \left( \frac{\gamma|q_K + q_0|}{v_0} \right)
\]

\[
G(t, s) = 2\sqrt{2} \frac{e^{-t/8}}{t^{1/2} \pi} \int_{s}^{+\infty} \sqrt{\cosh(u) - \cosh(s)} u e^{-u^2/2t} ds
\]

\[
\Phi(s) = 2 \arctan \left( \frac{\sinh^2(s) - \sinh^2(\text{lb})}{\sinh^2(\text{ub}) - \sinh^2(s)} \right)
\]

\[
\Psi(s) = 2 \text{artanh} \left( \frac{\sinh^2(s) - \sinh^2(\text{ub})}{\sinh^2(s) - \sinh^2(\text{lb})} \right)
\]

\[
q_0 = \frac{f^{1-\beta}}{1 - \beta}, \quad q_K = \frac{K^{1-\beta}}{1 - \beta}, \quad \nu = \frac{1}{2(1 - \beta)}
\]
SABR – Integration Formulas (Antonov et al.)

1D Integration 2013 but zero correlation

\[
C(K, T) = (f - K)^+ + \mathcal{O}(T, K)
\]

\[
\mathcal{O}(T, K) = \frac{T^{3/2}}{2\sqrt{2}} \exp \left( - \frac{1}{2} \frac{s_m^2}{\gamma^2 T} - \log \left( \frac{s_m^2}{2\gamma^2} \right) \right)
\]

\[
\log \left( K^{\beta \sqrt{v_0 \nu_m - A_m}} \right).
\]

\[
G(t, s) \approx \frac{\sinh(s)}{s} e^{-\frac{s^2}{2t} - \frac{t}{8} (R(t, s) + \delta R(t, s))}
\]

\[
R(t, s) = 1 + \frac{3t (s \coth(s) - 1)}{8s^2} - \frac{5t^2 \left( -8s^2 + 3 (s \coth(s) - 1)^2 + 24 (s \coth(s) - 1) \right)}{128s^4}
\]

\[
\frac{35t^3 \left( -40s^2 + 3 (s \coth(s) - 1)^3 + 24 (s \coth(s) - 1)^2 + 120 (s \coth(s) - 1) \right)}{1024s^6}
\]

\[
\delta R(t, s) = e^{t/8} \left( \frac{3072 + 384t + 24t^2 + t^3}{3072} \right)
\]
SABR – Antonov et al. Integration

Example

<table>
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<tr>
<th>Set</th>
<th>$F$</th>
<th>$T$</th>
<th>$\gamma$</th>
<th>$\beta$</th>
<th>$\nu_0$</th>
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<td>0.7</td>
<td>0.5</td>
<td>0.35</td>
<td>$-0.3, 0, 0.3$</td>
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<tr>
<td>Set 2</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>0.25</td>
<td>0.35</td>
<td>$-0.3, 0, 0.3$</td>
</tr>
</tbody>
</table>
SABR – Antonov et al. Integration Example
SABR – Integration Formulas (Antonov et al.)

General Case

• Projection onto a Bachelier, CEV or Zero Correlation SABR model
  \[ \tilde{\nu} = \tilde{\nu}^{(0)} + T\tilde{\nu}^{(1)} + \ldots \]

• The projection is not available for all values for the correlation

• No-arbitrage problems (negative values for the density), but much better than for the standard formulas

• The integration limits need to be adjusted wrt to the parameters

• Small values of the CEV parameter lead to wrong results

• Small values for the forward swap rates combined with small values of the CEV parameter can lead to wrong numbers very fast
The authors propose several models for mimicking the original model.

They suggest the zero correlation SABR with parameters $\tilde{\beta}, \tilde{\nu}, \tilde{\gamma}, \tilde{\rho}$

We have for the model parameters:

\[
\begin{align*}
\tilde{\beta} &= \beta \\
\tilde{\gamma} &= \sqrt{\gamma^2 - \frac{3}{2} (\rho^2 \gamma^2 + \nu \gamma \rho (1 - \beta) F^{\beta - 1})} \\
\tilde{\nu}^{(0)} &= \frac{2 \theta \Delta \tilde{\gamma}}{\theta^2 - 1} \\
\tilde{\nu}^{(1)}|_{K=F} &= \frac{1}{12} \left( 1 - \frac{\tilde{\gamma}^2}{\gamma^2} - \frac{3}{2} \rho^2 \right) \gamma^2 + \frac{1}{4} \beta \rho \nu \gamma F^{\beta - 1} \\
\Delta &= \frac{K^{1-\beta} - F^{1-\beta}}{1 - \beta} \\
\theta &= \left( \frac{\nu_{\min} + \rho \nu + \gamma \Delta}{(1 + \rho) \nu} \right)^{\frac{\tilde{\gamma}}{\gamma}} \\
\nu_{\min} &= \sqrt{\gamma^2 \Delta^2 + 2 \rho \gamma \Delta \nu + \nu^2}
\end{align*}
\]
Normal SABR – Integration Formulas (Korn et al.)

2D Integration \( C(F) = 1 \) but non-zero correlation

\[
C(K, T) = (f_0 - K)^+ + \frac{\sqrt{2}}{\nu \sqrt{1 - \rho^2}} \int_b^\infty g(b) h(b) db
\]

\[
h(b) = \int_{a_{min}}^{a_{max}} \frac{1}{\sqrt{\cosh(b) - \cosh(d(a))}} da
\]

\[
d(a) = \cosh^{-1} \left( 1 + \frac{N}{2(1 - \rho^2 v_0 a)} \right)
\]

\[
a_{min} = \frac{P - \sqrt{Q}}{2}; \quad a_{max} = \frac{P + \sqrt{Q}}{2}
\]

\[
N = (-\gamma m - \rho a + \rho v_0)^2 + (1 - \rho^2)(a - v_0)^2
\]

\[
P = -2m\nu\rho + 2v_0\rho^2 + 2v_0 \cosh(b) - 2v_0\rho^2 \cosh(b)
\]

\[
Q = 4(-v_0^2 - m^2\gamma^2 + 2mv_0\nu\rho)
\]

\[
\quad + (-2m\gamma\rho + 2v_0\rho^2 + 2v_0 \cosh(b) - 2v_0\rho^2 \cosh(b))^2
\]

\[
m = f - K
\]
Normal SABR

Implied Bachelier Volatilities - alpha

SABR Smiles

- Normal SABR

- Normal SABR

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Normal SABR

Implied Bachelier Volatilities - rho

![SABR Smiles Graph](image1)

![SABR Smiles Graph](image2)
Normal SABR

Implied Bachelier Volatilities - nu

SABR Smiles

- Implied volatility (4.40%)
- Strike (absolute): 2.94% 0.05 0.508

SABR Smiles

- Implied volatility (4.40%)
- Strike (absolute): 2.94% 0.05 0.38
Normal SABR – Integration Formula (Antonov)

1D Integration 2015 and non-zero correlation

\[ C(K, T) = (f - K)^+ + \mathcal{O}(T, K) \]

\[ \mathcal{O}(T, K) = \frac{V_0}{\pi} \int_{s_0}^{\infty} \frac{G(\gamma^2 T, s)}{\sinh(s)} \sqrt{\sinh^2(s) - (k - \rho \cosh(s))^2} \, ds \]

\[ \cosh(s_0) = \frac{-\rho k + \sqrt{k^2 + \bar{\rho}^2}}{\bar{\rho}^2} \]

\[ k = \frac{K - F_0}{V_0} + \rho \]

\[ V_0 = \frac{v_0}{\gamma} \]

\[ \bar{\rho} = 1 - \rho^2 \]
No-Arb SABR – PDE Ansatz
SABR becomes technical

- PDE method for calculating the SABR density numerically

- Efficient schemes (DO NOT USE Crank-Nicolson)

- Non-Standard grids and other tricks speed up the calculation

- Density using the PDE and option prices via numerical integration

- (New) approximation formulas for calibration

- Implementation details
No-Arb SABR – PDE Ansatz

The PDE

- First, we consider the Forward Equation for the reduced density

\[ \mathbb{P}[F < F_T < F + dF | F_t = f, v_t = v_0] \]

- This Forward Equation for the density is then given by

\[
\frac{\partial q}{\partial T} = \frac{v^2}{2} \frac{\partial^2}{\partial F^2} \left[ (1 + 2\rho\nu y + \nu^2 y^2) \exp(\rho\nu\nu\Gamma(F)(T - t)) C^2(F)q \right]
\]

\[ y(F) = \int_f^F \frac{dg}{C(g)} \quad \Gamma(F) = \frac{C(F) - C(f)}{F - f} \]

- By introducing a function \( D \) this simplifies to

\[
\frac{\partial q}{\partial T} = \frac{v^2}{2} \frac{\partial^2}{\partial F^2} \left[ D^2(F)q \right]
\]
The Boundary Conditions I

- Conservation (Probability Mass is equal to 1)

\[ q^L + \int_{F_{\text{Min}}}^{F_{\text{Max}}} q dF + q^R = 1 \]

leads to consider

\[ 0 = \frac{d}{dT} \left( q^L + \int_{F_{\text{Min}}}^{F_{\text{Max}}} q dF + q^R \right) \]

and finally requires

\[ \frac{dq^L}{dT} = \lim_{F \downarrow F_{\text{Min}}} \frac{v}{2} \left[ D^2(F)q \right]_F \]

\[ \frac{dq^R}{dT} = -\lim_{F \uparrow F_{\text{Max}}} \frac{v}{2} \left[ D^2(F)q \right]_F \]

\[ q^L(0) = 0 \]

\[ q^R(0) = 0 \]
No-Arb SABR – PDE Ansatz
The Boundary Conditions II

• Martingale Property (The forward is preserved)

$$\mathbb{E} [F_T | F_t = f, v_t = v_0] = F_{\text{Min}} q^L(T) + \int_{F_{\text{Min}}}^{\infty} F q DF + F_{\text{Max}} q^R(T) = f$$

$$0 = \frac{d}{dT} \left( q^L + \int_{F_{\text{Min}}}^{F_{\text{Max}}} q dF + q^R \right)$$

leads to the observation that

$$D^2(F)q \to 0, \quad F \downarrow F_{\text{Min}} \quad \quad D^2(F)q \to 0, \quad F \uparrow F_{\text{Max}}$$
No-Arb SABR – PDE Ansatz

The Final PDE

Continuous part of the distribution

$$\frac{\partial q}{\partial T} = \frac{\nu^2}{2} \frac{\partial^2}{\partial F^2} \left[ D^2(F)q \right] \quad q(0) = \delta(F - f)$$

Lower boundary (from model or the modellers decision)

$$\frac{dq^L}{dT} = \lim_{F \downarrow F_{\text{Min}}} \frac{\nu}{2} \left[ D^2(F)q \right]_F \quad q^L(0) = 0$$

Upper boundary (from discretization)

$$\frac{dq^R}{dT} = -\lim_{F \uparrow F_{\text{Max}}} \frac{\nu}{2} \left[ D^2(F)q \right]_F \quad q^R(0) = 0$$
No-Arb SABR – PDE Ansatz

Finite Difference Grid

\[ D^2(F)q \rightarrow 0; \quad F \uparrow F_{\text{Max}} \]

\[ P^R \delta(F - F_{\text{Max}}) \]

\[ q \{ F_{\text{Min}} < F < F_{\text{Max}} \} \]

\[ P^L \delta(F - F_{\text{Max}}) \]

\[ D^2(F)q \rightarrow 0; \quad F \downarrow F_{\text{Min}} \]
No-Arb SABR – PDE Ansatz

SABR becomes technical

- Implicit is slow but accurate
- Explicit is fast but may not suitable
- Mixing Implicit and Explicit using the Theta-Scheme (1/2 for Crank-Nicolson)
- How to choose lower and upper values for determining the grid

Standard schemes are tempting to apply but often more efficient/sophisticated methods are available.
No-Arb SABR – PDE Ansatz
Details – Transforming Variables (gain efficiency)

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No-Arb SABR – PDE Ansatz

Discretization

\[
\begin{align*}
    z_j &= z^- + jh \\
    y_j &= y(z_j - \frac{h}{2}) \\
    F_j &= F(y_j) \\
    C_j &= D(F_j) \\
    \Gamma_j &= \frac{F_j^\beta - f^\beta}{F_j - f} \\
    E_j(T) &= \exp(\rho \nu \alpha \Gamma_j T) \\
    t_n &= nT/N \\
    q^n_j &= q(z_j, t_n)
\end{align*}
\]

\[j = 1, \ldots, J, \quad n = 0, \ldots, N - 1\]
No-Arb SABR – PDE Ansatz

The Discrete Operator

\[ L_j^n q(z_j, t_n) = \frac{1}{\Delta} \frac{C_{j-1}}{F_j - F_{j-1}} E_{j-1}(t_n) q(z_{j-1}, t_n) \]

\[ - \frac{1}{\Delta} \left( \frac{C_j}{F_{j+1} - F_j} + \frac{C_j}{F_j - F_{j-1}} \right) E_j(t_n) q(z_j, t_n) \]

\[ + \frac{1}{\Delta} \frac{C_{j+1}}{F_{j+1} - F_j} E_{j+1}(t_n) q(z_{j+1}, t_n) \]

See LeFloch (2014)
No-Arb SABR – PDE Ansatz

Probabilities at the Boundaries

\[
\frac{C_0}{F_1 - F_0} E_0(T) q(t_0, T) = - \frac{C_1}{F_1 - F_0} E_1(T) q(z_1, T)
\]

\[
\frac{C_{J+1}}{F_{J+1} - F_J} E_{J+1}(T) q(t_{J+1}, T) = - \frac{C_J}{F_{J+1} - F_J} E_J(T) q(z_J, T)
\]
Problems with Crank-Nicolson

No-Arb SABR – Efficient Schemes
No-Arb SABR – Efficient Schemes
And with an efficient scheme

10 time steps 80 time steps 100 time steps 120 time steps
No-Arb SABR – PDE Ansatz

Standard Schemes

Implicit Euler

\[ q_j^{n+1} - q_j^n = \Delta q_j^{n+1} \]

\[ P^L(t_{n+1}) - P^L(t_n) = \Delta \frac{C_1}{F_1 - F_0} E_1(t_{n+1})q_1^{n+1} \]

\[ P^R(t_{n+1}) - P^R(t_n) = \Delta \frac{C_J}{F_{J+1} - F_J} E_J(t_{n+1})q_J^{n+1} \]

Crank Nicolson

\[ q_j^{n+1} - q_j^n = \Delta \left( \frac{L_j^{n+1}q_j^{n+1} + L_j^nq_j^n}{2} \right) \]

\[ P^L(t_{n+1}) - P^L(t_n) = \Delta \frac{C_1}{F_1 - F_0} \left( E_1(t_{n+1})q_1^{n+1} + E_1(t_n)q_1^n \right) \]

\[ P^R(t_{n+1}) - P^R(t_n) = \Delta \frac{C_J}{F_{J+1} - F_J} \left( E_J(t_{n+1})q_J^{n+1} + E_J(t_n)q_J^n \right) \]
No-Arb SABR – PDE Ansatz

Efficient Schemes, e.g. Lawson Swayne

\[ b = 1 - \frac{\sqrt{2}}{2} \]

\[ q_j^{n+b} - q_j^n = b \Delta L_j^{n+b} q_j^{n+b} \]

\[ P^L(t_{n+b}) - P^L(t_n) = b \Delta \frac{C_1}{F_1 - F_0} E_1(t_{n+b}) q_1^{n+b} \]

\[ P^R(t_{n+b}) - P^R(t_n) = b \Delta \frac{C_J}{F_{J+1} - F_J} E_J(t_{n+b}) q_J^{n+b} \]

\[ q_j^{n+2b} - q_j^{n+b} = b \Delta L_j^{n+2b} q_j^{n+2b} \]

\[ P^L(t_{n+2b}) - P^L(t_{n+b}) = b \Delta \frac{C_1}{F_1 - F_0} E_1(t_{n+2b}) q_1^{n+2b} \]

\[ P^R(t_{n+2b}) - P^R(t_{n+b}) = b \Delta \frac{C_J}{F_{J+1} - F_J} E_J(t_{n+2b}) q_J^{n+2b} \]
No-Arb SABR – PDE Ansatz
Efficient Schemes

Lawson-Swayne \( b = 1 - \frac{\sqrt{2}}{2} \)

\[
q_j^{n+1} = (\sqrt{2} + 1)q_j^{n+2b} - \sqrt{2}q_j^{n+b}
\]

\[
P^L(t_{n+1}) = (\sqrt{2} + 1)P^L(t_{n+2b}) - \sqrt{2}P^L(t_{n+b})
\]

\[
P^R(t_{n+1}) = (\sqrt{2} + 1)P^R(t_{n+2b}) - \sqrt{2}P^R(t_{n+b})
\]
No-Arb SABR – Pricing via Integration

And with an efficient scheme

Integrating with respect to the density (blue)

- Choose an integration scheme
- Just use the representation of the PDE with appropriate handling of the left tail (red)
No-Arb SABR – Pricing via Integration

Option Pricing Formulas

We assume that the FDM Scheme applied uses \( N \) steps and a size of \( \Delta \)

\[
F_{\text{Min}} + (k - 1)\Delta < K < F_{\text{Min}} + k\Delta
\]

\[
V_{\text{Call}}(T, K) = \frac{1}{2} \left[ F_{\text{Min}} + k\Delta - K \right]^2 q_k^N + \sum_{j=k+1}^{N} \left[ F_{\text{Min}} + \left( j - \frac{1}{2} \right)\Delta - K \right] \Delta q_j^N + [F_{\text{Max}} - K] q^R
\]

\[
V_{\text{Put}}(T, K) = \frac{1}{2} \left[ K - F_{\text{Min}} - (k - 1)\Delta - K \right]^2 q_k^N + \sum_{j=1}^{k} \left[ K - F_{\text{Min}} - \left( j - \frac{1}{2} \right)\Delta \right] \Delta q_j^N + [K - F_{\text{Min}}] q^L
\]
No-Arb SABR – PDE Ansatz
SABR becomes technical
No-Arb SABR – Approximation Formulas
Old Results from Libor Market Models

• Andersen/Brotherton-Ratcliffe show an approximation in the context of Libor Market Models with Stochastic Volatility
• The results can be carried over to the SABR model

\[
\sigma(T, K) = \frac{\sum_0(K)u^{1/2}(T) + \sum_1(K)u^{3/2}(T)}{\sqrt{T}}
\]

\[
u(T) := T + \frac{1}{2}\rho\nu\alpha\Gamma(K)T^2 + O(T^3)
\]

\[
\Gamma(K) = \frac{(K + b)^\beta - (F + b)^\beta}{K - F}
\]

• This approximation can be used in calibration
No-Arb SABR – Approximation Formulas

Old Results from Libor Market Models

LogNormal Volatility Approximation

\[ \sigma_{LN}^{AB} = \frac{1}{(\xi(K))} \log \left( \frac{f + b}{K + b} \right) \left[ 1 + \left( g_{LN}(K) + \frac{1}{4} \rho \nu \alpha \Gamma(K) \right) T \right] \]

\[ g_{LN}(K) = -\frac{1}{x(\xi(K))^2} \log \left( \frac{\log \left( \frac{f + b}{x(\xi(K))} \right)}{x(\xi(K))} \sqrt{\frac{(f + b)(K + b)}{D(f)D(K)}} \right) \]

\[ D(K) = \sqrt{\alpha^2 + 2\alpha \rho \nu y(K) + \nu^2 y(K)^2 K^\beta} \]

\[ \Gamma(K) = \frac{(K + b)^{\beta} - (f + b)^{\beta}}{K - f} \]

\[ y(K) = \frac{(K + b)^{1-\beta} - (f + b)^{1-\beta}}{1 - \beta} \]

Bachelier Volatility Approximation

\[ \sigma_N^{AB} = \frac{f - K}{(\xi(K))} \left[ 1 + \left( g_N(K) + \frac{1}{4} \rho \nu \alpha \Gamma(K) \right) T \right] \]

\[ g_N(K) = -\frac{1}{x(\xi(K))^2} \log \left( \frac{f - K}{x(\xi(K)) \sqrt{D(f)D(K)}} \right) \]

\[ x(\xi(K)) = \frac{1}{\nu} \log \left( \frac{\sqrt{1 - 2\rho \xi(K) + \xi(K)^2} - \rho + \xi(K)}{1 - \rho} \right) \]

\[ \xi(K) = \frac{\nu}{\alpha(1 - \beta)} \left[ (f + b)^{1-\beta} - (K + b)^{1-\beta} \right] \]
Comparison – Approximation vs Integration/PDE

Benchmarking Approximation Formulas

- Comparison of approximation formulas
  - Comparison to Bachelier volatility
  - Comparison to Call prices

- Benchmark against Integration/PDE Solutions
  - Integration/PDE leads to density and prices
  - Implied Bachelier Vol Solver is used (-> see later)
Comparison – Approximation Formulas for SABR

Hagan, Andersen/Brotherton-Ratcliffe

Approxformulas vs PDE Solution

\[ \begin{align*}
\text{forward} & \quad 1 \\
\beta & \quad 0.25 \\
v0/\alpha & \quad 0.35 \\
\gamma/\nu & \quad 1 \\
\rho & \quad 0.35 \\
\tau & \quad 2 \\
\text{displacement} & \quad 0
\end{align*} \]

Values from Hagan, P. ICBI Global Derivatives 2014

\[ \text{Bachelier Implied Volatility Comparison} \]

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Comparison – Normal Vol Approximation vs PDE

Hagan, Andersen/Brotherton-Ratcliffe

Approxformulas vs PDE Solution

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
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Comparison – Approximations H vs AB

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Comparison – Approximations H vs AB

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</tbody>
</table>

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Comparison – Approximations H vs AB

Conclusion

• Numerical experiments suggest that the correlation determines which approximation to apply (Hagan approximation or Andersen/Brotcherton-Ratcliffe approximation)

• The calculation time does not differ between the methods

• We need to validate the calibration procedure (Do the parameters run to the boundaries?)

• The approximation formulas lead to better results than the PDE pricer
  • PDE approach is numerically much more involved
  • We need methods to stabilize the set-up
Free Boundary SABR – (Antonov et al.)

SABR gets technical

There is an approximation formula which numerically approximates the free boundary SABR model with zero correlation:

\[ dS = \alpha|S|^\beta dW_1(t) \]
\[ d\alpha = \nu \alpha dW_2(t) \]
\[ \langle dW_1(t), dW_2(t) \rangle = \rho dt \]

For the general model a Markovian projection has to be applied to approximate this case by means of the zero correlation model.

We have a PDE solution and approximation formulas here as well, see

Free Boundary SABR - Example
Calculation via Numerical Integration
Free Boundary SABR – Approximation I

Approximation in Free SABR Setting

\[
\sigma_B^H(T, K) \approx \frac{\alpha(f - K)}{\int_K^f \frac{ds}{C(s)}} \left( \frac{\xi}{x(\xi)} \right) \left( 1 + \left( g\alpha + \frac{\rho\nu\alpha}{4} \frac{C(f)C(K)}{f - K} + \frac{2 - 3\rho^2}{24} \nu^2 \right) T \right)
\]

For the free SABR case we find:

\[
I := \int_K^f |x|^{-\beta} \, dx
\]

\[
I = \begin{cases} 
\frac{(-f)^{1-\beta} - (-K)^{1-\beta}}{1-\beta} & K < 0, f < 0 \\
\frac{f^{1-\beta} + (-K)^{1-\beta}}{1-\beta} & K < 0, f > 0 \\
\frac{f^{1-\beta} - K^{1-\beta}}{1-\beta} & K > 0, f > 0 
\end{cases}
\]
Free Boundary SABR – Approximation II
Approximation in Free SABR Setting

\[ \sigma^A_B(T, K) = \begin{cases} 
\frac{|f| + |K|}{\xi_{KK}} (1 + (g_K + \frac{1}{4} \rho \nu \alpha \Gamma_K) T) & K < 0, F \geq 0 \\
\frac{|f| - |K|}{\xi_{KK}} (1 + (g_K + \frac{1}{4} \rho \nu \alpha \Gamma_K) T') & \text{else}
\end{cases} \]

\[ K < 0 \]
\[
g_K = -\log \left( \frac{|f| + |K|}{\xi_{KK} \sqrt{DfDK}} \right) / \xi_{KK}^2
\]
\[
\Gamma_K = \frac{-|K| \beta - |f| \beta}{1 - \beta}
\]
\[
y_K = \frac{-|K|^{1-\beta} - |f|^{1-\beta}}{1 - \beta}
\]
\[
Df = \alpha |f| \beta
\]
\[
Dk = \sqrt{\alpha^2 + 2 \alpha \rho \nu y_K + \nu^2 y_K^2 |K| \beta}
\]
\[
\xi_K = \frac{\nu}{\alpha(1-\beta)} (|f|^{1-\beta} + |K|^{1-\beta})
\]
\[
\xi_{KK} = \frac{\log(\sqrt{1 - 2 \rho \xi_K + \xi_K^2} + \rho + \xi_K)}{\nu(1-\rho)}
\]

\[ K \geq 0 \]
\[
g_K = -\log \left( \frac{|f| - |K|}{\xi_{KK} \sqrt{DfDK}} \right) / \xi_{KK}^2
\]
\[
\Gamma_K = \frac{|K| \beta - |f| \beta}{1 - \beta}
\]
\[
y_K = \frac{|K|^{1-\beta} - |f|^{1-\beta}}{1 - \beta}
\]
\[
Df = \alpha |f| \beta
\]
\[
Dk = \sqrt{\alpha^2 + 2 \alpha \rho \nu y_K + \nu^2 y_K^2 |K| \beta}
\]
\[
\xi_K = \frac{\nu}{\alpha(1-\beta)} (|f|^{1-\beta} - |K|^{1-\beta})
\]
\[
\xi_{KK} = \frac{\log(\sqrt{1 - 2 \rho \xi_K + \xi_K^2} - \rho + \xi_K)}{\nu(1-\rho)}
\]
Free Boundary SABR - Example
Comparison Integration, PDE, Approximation
Free Boundary SABR - Example
Comparison Integration, PDE, Approximation
Free Boundary SABR - Example
Comparison Integration, PDE, Approximation
Free Boundary SABR - Example
Comparison Integration, PDE, Approximation
Free Boundary SABR - Example

Calculation via PDE

• PDE method for calculating the SABR density numerically can be adapted to the Free SABR case

• Non-Standard grids and other tricks speed up the calculation again

• Density using the PDE and option prices via numerical integration

• Being able to check the integration and the (new) approximation formulas
Free Boundary SABR - Example
Calculation via PDE

Transformation of Variables
Solving the PDE

<table>
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<th>Value</th>
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<tr>
<td>nd</td>
<td>4</td>
</tr>
</tbody>
</table>
Free Boundary SABR – (Antonov et al.)

SABR gets technical

- Integrals solution for correlation 0 case and Markovian projection for other cases (might be unstable for large values of absolute correlation)
- PDE approach as shown for No-Arbitrage SABR
- Approximation formulas as shown for No-Arbitrage SABR

This allows to test approximation formulas and fastens the calibration.

Are the good old times back?
Free Boundary SABR – (Antonov et al.)

Examples

We have applied the modified PDE solution to different parameter sets. The following illustrations are for different values of $T$ and $\beta$. 
Free SABR
Bachelier Implied Volatility - beta

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## Free SABR

Bachelier Implied Volatility - rho

<table>
<thead>
<tr>
<th>Strike (absolute)</th>
<th>Bachelier Implied Volatility</th>
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</thead>
<tbody>
<tr>
<td>4.70%</td>
<td>0.18</td>
</tr>
<tr>
<td>-0.05</td>
<td>0.384</td>
</tr>
<tr>
<td>4.70%</td>
<td>0.185</td>
</tr>
<tr>
<td>-0.13</td>
<td>0.384</td>
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</tbody>
</table>

SABR Smiles

![SABR Smiles](image1.png)

SABR Smiles

![SABR Smiles](image2.png)

SABR Smiles

![SABR Smiles](image3.png)
Free SABR
Bachelier Implied Volatility - \( nu \)
Free SABR
Bachelier Implied Volatility - alpha
Free Boundary SABR – (Antonov et al.)
Calibrating Caps with Free Boundary SABR

Calibration example on EUR June data.
Free Boundary SABR – (Antonov et al.)
Calibrating Caps with Free Boundary SABR
Free Boundary SABR – (Antonov et al.)
Calibrating Caps with Free Boundary SABR
SABR
Shape of the (Bachelier) Smile

DD SABR / No Arb SABR
Implied Volatilities tend decrease for small strikes (‘SABR Knee’)  

Free SABR / Normal SABR
Implied Volatilities do not decrease for small strikes
SABR – Sticky Absorbing SABR
A New Parametrization

The SABR model was:

\[ dF(t) = \nu(t)C(F(t))dW_1(t) \]
\[ d\nu(t) = \gamma \nu(t)dW_2(t) \]
\[ \langle dW_1(t), dW_2(t) \rangle = \rho dt \]
\[ F(0) = f \]
\[ \nu(0) = \nu_0 \]

To be able to handle negative rates we may choose:

\[ C(F) = (F + a)^\beta \quad \text{Displaced SABR} \]
\[ C(F) = 1 \quad \text{Normal SABR} \]
\[ C(F) = |F|^\beta \quad \text{Free SABR} \]

\[ C(F) = |F + a|^\beta \]
\[ F_{\text{Min}} = -b \]

Sticky Absorbing SABR
SABR – Sticky Absorbing SABR
Recovering the SABR Knee (using $F_{\text{Min}} = -b$)
SABR – Sticky Absorbing SABR
Controlling the Stickyness of Rates (using

\[ C(F) = |F + a|^{\beta} \]
SABR

Summary

• Using new numerical methods the SABR model can be tackled

• Integration, PDE and Approximation formulas are available

• The probability density and prices can be calculated efficiently

• The Free SABR model, the Displaced and the Normal SABR can accommodate negative rates

• The Free SABR and Normal SABR imply that the implied volatilities do not tend to 0 for very small number

• To obtain this result DD SABR or an artificial lower bound should be applied
Mixture of Models
- ZABR, SABR -
Mixture of Models
Rationale – Controlling the Wings

• Use different models / different parametrizations which are analytically tractable

• Take a convex combination

• This was first done with Shifted Log-Normal models (different displacements)

• We can apply the ZABR or the (Free Boundary) SABR model

• This methodology is for pricing Vanillas since it does not give a reasonable model dynamics -> See Piterbarg „Mixture of Models: A Simple Recipe for a … hangover“ (2009)
ZABR – Mixture of ZABR Models
Smile Control by Mixing?

Mixing ZABR models to control the behaviour of the wings.
This allows to control the behaviour of the left or the right wing separately.
In contrast to simply using ZABR.

Figure: ZABR implied log-normal (left) and normal volatilities (right) for $k = 0.5, 1, 1.5, 1.7$ and the parameters from $\omega = 0.0873$, $\beta = 0.7$, $\gamma = -0.46$, $\eta = 0.47$, $f = 0.0325$ and $T = 10$. 
The same mixing approach can be applied to SABR models. In this case Antonov et al. suggest to mix free SABR with zero correlation SABR. However, using their approach prices have to be calculated and transformed to implied Bachelier volatilities. The approach recently put forward by Kienitz only considers implied Bachelier volatilities.
Here we consider the mixing of SABR models while keeping the tail behaviour to the right the same for both SABR models.

We use the approximation formula in the Free SABR context.
Negative Rates, SABR, CMS - Literature

Papers

- Andreasen, J., Huge, B., „ZABR Expansion for the Masses“, SSRN 2012
- Antonov, A., Spector, M., „Advanced Analytics for the SABR Model“, SSRN 2012
- Antonov, A., „The Free Boundary SABR: Extensions to Negative Rates“, SSRN
- Kienitz, J., „Pricing CMS Spread Options“, Global Derivatives 2011
- LeFloch, F. „Finite Difference Techniques for Arbitrage Free SABR“, SSRN 2014
- LeFloch, F. „Explicit SABR Calibration through simple Expansions“, SSRN 2014

Books

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